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THE LOGARITHMIC FINITE ELEMENT METHOD

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Abstract. The Logarithmic finite element (LogFE) method extends the Ritz-Galerkin method to approximations on a non-linear finite-dimensional manifold in the infinite-dimensional solution space. Formulating the interpolant on the logarithmic space allows for a novel treatment of the rotational component of the deformation, and induces a strong coupling between rotations and translations. The Logarithmic finite element method provides a transformation of the initial configuration that is not restricted to an isoparametric formulation.

1 Introduction

In the Ritz-Galerkin method, the configuration of a material body is fully determined by the translations of its material points (or the generalized coordinates of the points of a representation of the body) relative to the initial configuration. As a result, rotational field values are usually given as elements of a subspace of the state space that is completely independent from the subspace of translations. Models based on rotational pseudo-vectors, as well as models based on the special orthogonal group of rotations, $SO(n,\mathbb{R})$, both share these characteristics. For an overview of geometrically exact beam formulations, involving different approaches to modeling rotations, see [8, 9, 12].

The Logarithmic finite element (LogFE) method [10, 11, 12] extends the Ritz-Galerkin method to approximations on a non-linear finite-dimensional manifold in the infinite-dimensional solution space. This manifold can be constructed so that it contains both translations and rotations. As a result, it is possible to induce a strong coupling of the rotational and the translational component of the deformation of a body. V. Sonneville, A. Cardona, and O. Brüls [14] propose a geometrically exact beam formulation that maps the centerline of the beam into the special Euclidean group, $SE(n,\mathbb{R}) = SO(n,\mathbb{R}) \ltimes \mathbb{R}^n$. By contrast, the Logarithmic finite element method formulates a transformation, i.e. a map of the initial configuration to the current configuration. However, the special

Euclidean group, which is a semidirect product, turns out to be unsuitable for defining such a transformation. We therefore construct a Lie group based on a direct product of rotations and transformations in order to formulate the transformation of the initial configuration.

In section 2, we briefly touch on the notion of the constraint manifold on the configuration space, which is central to formulations of structural elements such as beams. Section 3 provides some basic results of the theory of Lie groups, and may also help to identify those aspects of Lie group theory in the literature that are relevant to the Logarithmic finite element method. We encourage readers unfamiliar with the mathematical background to consult the introductions to Lie group theory referenced in this section. Section 4 describes how a transformation of the initial configuration is defined on a Lie algebra and highlights the importance of constructing the Lie algebra in way that ensures the independence of the impact of the degrees of freedom at the nodes of the element. Section 5 explains how the Logarithmic finite element method extends the Ritz-Galerkin method. In section 6, we construct a specific Lie algebra as part of the formulation of beam models based on Bernoulli as well a Timoshenko kinematics and present the action of its associated Lie group on the initial configuration of a beam. Section 7 concludes the exposition with some general observations.

2 Configurations and constraint manifolds

Following C. Truesdell and W. Noll [15], we define the material body $\mathcal{B}$ as a compact m-dimensional differentiable manifold endowed with a finite measure μ. A material point of the body is denoted P. The physical space $\mathcal{S}$ is defined as an n-dimensional differentiable manifold, with $n \geq m$. A configuration $\kappa \in C^1(\mathcal{B},\mathcal{S})$ is given as a differential embedding of the body into the physical space. The configuration manifold $\mathcal{Q}$ is given as the set of all configurations.[1]

The *constraint manifold* $\mathcal{C}$ comprises all admissible configurations of the material body [2]. For example, the configuration $\kappa \in \mathcal{Q}$ of a three-dimensional beam in a three dimensional physical space that is torsion-free and whose centerline remains in a given plane in the physical space is fully determined by the configuration $\tilde{\kappa} \in \tilde{\mathcal{Q}}$ of its one-dimensional centerline $\tilde{\mathcal{B}}$ in a two-dimensional Euclidean space $\mathbb{E}^2$. $\tilde{\kappa}$ may be given as the map of a set of points representing the body to a space $\tilde{\mathcal{S}}$ of generalized coordinates that are not restricted to actual spatial positions in a physical space. It may, for example, include the orientation of the cross-sections of a beam as generalized coordinates. The embedding $\iota : \tilde{\mathcal{Q}} \to \mathcal{Q}$ defines the admissible configurations of the beam.

In the Logarithmic finite element method, the approximation space is generally constructed as a finite-dimensional submanifold $\mathcal{C}_h$ of the constraint manifold $\mathcal{C}$.

3 Lie groups and Lie algebras

Throughout this text, we follow the notational conventions of the academic field in which the respective knowledge emerged. Therefore, while we seek global consistency

[1] For the notation, see also [2].

in the notation related to mechanics, mathematical background will be presented using scoped notation. For accessible introductions to Lie group theory, see [3, 4, 5, 7].

3.1 Basic concepts

A *Lie group* G is a differential manifold endowed with a differentiable operation $\varphi : G \to \mathrm{Aut}(G)$, $g \mapsto \varphi_g$, such that $\varphi_g : G \to G$, $h \mapsto \varphi_g(h)$. Thus, the *group product* is given as $G \times G \to G$, $(g,h) \mapsto gh := \varphi_g(h)$. We denote the *identity element* of G as e_G, φ_{e_G} is the identity map on G. Note that, for a translational Lie group T, with $s,t \in T$, we have $st := \varphi_s(t) = s + t$, and therefore $e_T = 0$. A Lie group may be endowed with a group action σ on a manifold M such that for $g,h \in G$, $m \in M$, $\sigma_g \circ \sigma_h(m) = \sigma_{gh}(m)$. A *matrix Lie group* is isomorphic to a closed subgroup of some general linear group $GL(n,\mathbb{K})$, with $n \in \mathbb{N}$, $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$.

The tangent space of G at the identity element, T_eG, is a real vector space. For $X \in T_eG$, let $\gamma_X : \mathbb{R} \to G$ define an integral curve γ_X such that $\gamma_X(0) = e_G$, $\gamma_X(s+t) = \gamma_X(s)\gamma_X(t)$ and $\partial_t \gamma_X(t)\big|_{t=0} = X$. Then,

$$\exp_G(tX) := \gamma_X(t)\,. \tag{3.1}$$

The group product on G gives rise to the *adjoint representation* $\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$. For $Y \in T_eG$, $g \in G$, let $\exp \circ \mathrm{Ad}_g(Y) := g \exp(Y) g^{-1}$, which implies $\mathrm{Ad}_g(Y) := gYg^{-1}$ for matrix Lie groups. Then,

$$\mathrm{ad}_X = \partial_t \mathrm{Ad}_{\exp(tX)}\big|_{t=0}\,. \tag{3.2}$$

ad and ad_X are linear functions. The vector space T_eG, endowed with the adjoint representation, is referred to as the *Lie algebra* $\mathfrak{L}(G)$ associated with the Lie group G, denoted as $\mathfrak{g}$ hereafter. The *Lie bracket*, also referred to as the commutator, on $\mathfrak{g}$ is defined as $[X,Y] := \mathrm{ad}_X(Y)$. If G is a matrix Lie group, the multiplication on $M_n(\mathbb{K})$ can be used, and

$$[X,Y] = XY - YX\,. \tag{3.3}$$

We denote the vector space of a Lie algebra $\mathfrak{g}$ as $V_{\mathfrak{g}}$. Thus, $V_{\mathfrak{L}(G)} = T_eG$.

A Lie group G is called *abelian* if its elements commute, i.e., for every $g,h \in G$, $gh = hg$. A Lie algebra $\mathfrak{g}$ is called abelian if, for every $X,Y \in \mathfrak{g}$, $\mathrm{ad}_X(Y) = 0$. A Lie algebra $\mathfrak{g}$ is abelian if, and only if, the connected Lie group $G^0 = \langle \exp \mathfrak{g} \rangle$ containing the identity element is abelian. All Lie groups considered in the following exposition are connected.

A *subalgebra* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ such that $[X,Y]_{\mathfrak{g}} \in \mathfrak{h}$ for all $X,Y \in \mathfrak{h}$. A subspace U of $\mathfrak{g}$, equipped with the Lie bracket on $\mathfrak{g}$, generates the subalgebra $\langle U \rangle_{\mathfrak{g}} \leq \mathfrak{g}$, the smallest subalgebra of $\mathfrak{g}$ that contains U.

3.2 Direct and semidirect products

The *semidirect product* $G = N \rtimes_\alpha H$ of two Lie groups N and H, with $N \times \{e_H\}$ being a normal subgroup of G, is the manifold $N \times H$, endowed with the product

$$G \times G \to G,\ \Big((n,h),(n',h')\Big) \mapsto \Big(n\alpha\big(h,n'\big),hh'\Big)\,. \tag{3.4}$$

The Lie algebra $\mathfrak{L}(G) = \mathfrak{n} \rtimes_\beta \mathfrak{h}$, subsequently referred to as $\mathfrak{g}$, of the semidirect product G is given by the direct sum of the vector spaces $V_\mathfrak{g} \oplus V_\mathfrak{h}$, endowed with the Lie bracket

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \Big((Y,X), (Y',X') \Big) \mapsto \Big(\beta\big(X,Y'\big) - \beta\big(X',Y\big) + \big[Y,Y'\big], \big[X,X'\big] \Big). \tag{3.5}$$

With the isomorphism $\pi : \mathfrak{n} \times \{e_H\} \to \mathfrak{n}$, $(Y,0) \mapsto Y$ we have $\beta(X,Y) = \pi \circ \mathrm{ad}_{(0,X)}(Y,0)$ [5, pp. 308–9][6, pp. 102–4]. For Lie algebras based on matrix Lie groups, we obtain $\beta(X,Y) = \pi \circ \big((0,X)(Y,0) - (Y,0)(0,X)\big)$. Let N and H be closed subgroups of some general linear group $GL(n,\mathbb{K})$, let σ denote the group action of H on N, τ the action of $\mathfrak{h}$ on $\mathfrak{n}$, both derived from the matrix multiplication on $M_n(\mathbb{K})$. Then, for $\alpha(h,n') = \sigma_h(n')$, using (3.2), we have $\beta(X,Y) = \tau_X(Y)$.[2]

The *direct product* $G \times H$ and the *direct sum* $\mathfrak{g} \oplus \mathfrak{h}$ can be understood as special cases of the semidirect product and the semidirect sum, with $\alpha(h,n') = n'$, $\beta(X,Y) = 0$. In the notation of the direct product of Lie groups and the semidirect sum of Lie algebras, we omit the functions α and β.

The product operations of Lie groups that are subgroups of $GL(n,\mathbb{K})$ can be derived from the matrix multiplication on $M_n(\mathbb{K})$. Thus, for the semidirect product $N \rtimes_\alpha H$, with $N = \mathbb{K}^{n-1}$, $nn' = n + n'$ for $n \in N$, $H = GL(n-1,\mathbb{K})$, the group action σ of H on N given by the action of $GL(n-1,\mathbb{K})$ on $\mathbb{K}^{n-1}$, and $\alpha(h,n) = \sigma_h(n)$, we obtain

$$N \rtimes_\alpha H \cong \left\{ \begin{pmatrix} h & n \\ 0 & 1 \end{pmatrix} \in GL(n,\mathbb{K}) \,\middle|\, n \in N, h \in H \right\} < GL(n,\mathbb{K}). \tag{3.6}$$

3.3 The exponential function and its derivative

The *exponential map* $\exp_G : \mathfrak{g} \to G$, for a Lie group G, is given by $X \mapsto \gamma_X(1)$ (see equation (3.1)). We will omit the subscript if there is no ambiguity. From the theory of differential equations, we obtain the result [5] that, for matrix Lie algebras,

$$\gamma_X(1) = \exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k. \tag{3.7}$$

Note that, while the exponential function is not necessarily surjective onto G, it is locally diffeomorphic at 0, and $\exp_G(0) = e_G$.

The *logarithmic derivative* δ of the exponential map $\exp : \mathfrak{g} \to G$, discussed in [5], is given by

$$\delta(\exp) : \mathfrak{g} \to \mathrm{Aut}(\mathfrak{g}), X \mapsto \sum_{k=1}^{\infty} \frac{(-\mathrm{ad}_X)^{k-1}}{k!}. \tag{3.8}$$

The *derivative* of the exponential map is given as

$$D\exp : X \mapsto \exp(X)\,\delta(\exp)(X). \tag{3.9}$$

$D\exp(X) \in \mathrm{Diff}\big(\mathfrak{g}, T_{\exp(X)}G\big)$ is a linear function.

[2] In [12], subscripts in the notation of semidirect products and sums refer to functions α' and β', such that $\alpha(h,n) = \sigma_H\big(\alpha'(h)\big)(n)$, $\beta(X,Y) = \sigma_\mathfrak{h}\big(\beta'(X)\big)(Y)$, and the group actions σ_H and $\sigma_\mathfrak{h}$ are induced by the operations of $GL(n,K) \geq G$ and $\mathfrak{gl}(n,K) \geq \mathfrak{g}$.

4 Shape functions on the Logarithmic space

4.1 Transformations

In the following, all configurations refer to elements of the discretized configuration space $\mathcal{Q}_h$. We omit the respective subscripts for brevity.

A configuration κ associates a point in the physical space $\mathcal{S}$ to each material point of the body $\mathcal{B}$. It can thus be understood as a set in $\mathcal{B}\times\mathcal{S}$. Let $\pi:\mathcal{B}\times\mathcal{S}\to\mathcal{B}$, $(P,\mathbf{x})\mapsto P$. Then, a bundle morphism $\chi:\mathcal{Q}\supseteq Q\to\mathcal{Q}$, $(P,\mathbf{x})\mapsto(P,\mathbf{x}')$, satisfies $\pi\circ\chi(\kappa)=\pi(\kappa)$ for all $\kappa\in Q$. We refer to a bundle morphism of a configuration as a *transformation*. We denote the restriction of χ to a single finite element as χ^e. We assume that a single finite element can be parameterized by a chart (Ω^e,J^e) on the body $\mathcal{B}$. We identify the points of $\Omega_\square:=J^e(\Omega^e)$ by $\boldsymbol{\xi}$. Let $\mathcal{Q}^e:=\Omega_\square\times\mathcal{S}$ denote the configuration space of a finite element, $\pi^e:\Omega_\square\times\mathcal{S}\to\Omega_\square$, $(\boldsymbol{\xi},\mathbf{x})\mapsto\boldsymbol{\xi}$. Then, a bundle morphism $\chi^e:\mathcal{Q}^e\supseteq Q^e\to\mathcal{Q}^e$, $(\boldsymbol{\xi},\mathbf{x})\mapsto(\boldsymbol{\xi},\mathbf{x}')$, satisfies $\pi^e\circ\chi^e(\kappa^e)=\pi^e(\kappa^e)$ for all $\kappa^e\in Q^e$. In the following, we will define the transformation χ^e of a finite element, which, in conjunction with a given initial configuration κ_0^e, also defines its current configuration κ_τ^e.

Let U denote a subspace of a Lie algebra $\mathfrak{g}$. A basis in U is given by the vectors $\mathcal{V}=(\mathbf{v}_k)_{1\le k\le K}$. We associate one or more shape functions $N_{k,l}(\boldsymbol{\xi})$, $1\le l\le L$, with each basis vector $\mathbf{v}_k$. Each shape function is associated with a degree of freedom $u_{k,l}$ at the element level. In order to obtain a global finite system, suitable degrees of freedom at the element level may be linked to global degrees of freedom, while other degrees of freedom may remain as internal degrees of freedom. With these components, the transformation of a finite element is given by

$$\chi^e:(\boldsymbol{\xi},\mathbf{x})\mapsto\big(\boldsymbol{\xi},\sigma_{g(\boldsymbol{\xi})}\mathbf{x}\big)\,, \tag{4.1a}$$

with

$$g(\boldsymbol{\xi})=\exp\big(X(\boldsymbol{\xi})\big)\in G, \tag{4.1b}$$

$$X(\boldsymbol{\xi})=\sum_{\substack{1\le k\le K\\1\le l\le L}}u_{k,l}N_{k,l}(\boldsymbol{\xi})\,\mathbf{v}_k\in\mathfrak{g}=\mathfrak{L}(G)\,. \tag{4.1c}$$

σ is the action of the Lie group G on the physical space $\mathcal{S}$.

The concept of the transformation of a configuration can similarly be applied to configurations $\tilde{\kappa}\in\tilde{\mathcal{Q}}$, which identify elements of the constraint manifold $\mathcal{C}$. The beam models presented in section 6 will be based on transformations in $\tilde{\mathcal{Q}}$.

4.2 Degrees of freedom

In general, we aim to separate the impact of changes of different degrees of freedom on different components of the total deformation of a finite element as much as possible. Among other advantages, such a separation allows for the meaningful definition of global degrees of freedom. For example, given a material point $P\in\Gamma^{e,e'}=\Omega^e\cap\Omega^{e'}$, two degrees of freedom on the element-level in Ω^e and $\Omega^{e'}$, which define the translation of P in a given spatial direction, may be associated with a global degree of freedom governing the translation of P.

The degrees of freedom $u_{k,l}$ whose shape functions do not do not vanish of sufficient order at a given node I at $P \in \Gamma^{e,e'}$ impact the deformation of the finite element at that node in a significant way.[3] We denote the set of vectors $\mathbf{v}_k$ associated with these degrees of freedom as $\mathcal{V}_I$. The separation of the impact of degrees of freedom on the element level necessitates that the elements of the subspace $\langle \mathcal{V}_I \rangle$ belong to a subalgebra $\mathfrak{h} \leq \mathfrak{g} := \mathfrak{L}(G)$ that is given as the direct sum of its constituent Lie algebras, i.e. $\mathfrak{h} = \bigoplus_{m \in M} \mathfrak{h}_m$. This ensures that non-commutativity is restricted to sets of degrees of freedom that are related to intrinsically non-commutative deformations. For example, the impact of rotations in the Euclidean space is non-commutative, and sets of scalar degrees of freedom related to such rotations may be best understood as a single vector-valued degree of freedom.

As noted above, the span of the set of vectors $\mathcal{V}_I$ is a subspace of the vector space of the Lie algebra $\mathfrak{g}$. However, if endowed with the Lie bracked of $\mathfrak{g}$, it is not necessarily a subalgebra of $\mathfrak{g}$. While the transformation $g(\boldsymbol{\xi})$ itself can be formulated without making use of the Lie bracket on $\mathfrak{g}$, the derivatives of $g(\boldsymbol{\xi})$ with regard to $\boldsymbol{\xi}$ must be computed using the adjoint representation, as shown in section 3.3.

5 Extending the Ritz-Galerkin method

5.1 Non-linear transformations as a generalization of displacements

In the Ritz-Galerkin method, the transformation χ of the initial configuration κ_0 is restricted to displacements of the physical locations of material points. Thus, the group action of the Lie group G in equation (4.1a) on the spatial locations $\mathbf{x}_0$ of material points in the initial configuration must be restricted to a translation. Furthermore, in the Ritz-Galerkin method, the interpolant is given as a linear combination of the shape functions. However, the exponential function is generally non-linear. Thus, in order for equation (4.1a) to satisfy the requirements of the Ritz-Galerkin method, the Lie group G itself must be translational, i.e. its group operation must be given as $gh = \varphi_g(h) = g + h$ for $g, h \in G$.

With regard to the exact solution, denoted u, every deformation, independent of the formulation of the pointwise maps transforming the positions of material points in the physical space, can be expressed as a pointwise translation of the material points. The exact solution space $V \ni u$ can therefore be conceived as an infinite-dimensional space of displacements. We will retain this notion of the exact solution space in the discussion of the Logarithmic finite element method.

5.2 The approximation space as a non-linear submanifold

In the Ritz-Galerkin method, the approximation space V_h is given as a finite-dimensional linear subspace of V. Inter alia, this implies that, given deformations $u_h^*, u_h^{*\prime} \in V_h$, all linear combinations of these deformations are also elements of the approximation space V_h, and may be obtained by choosing an appropriate set of values for the degrees of freedom

[3] For a discussion of the requirements related to the order of zeros of different shape functions, see [12].

associated with the shape functions that generate V_h. The space of degrees of freedom thus maps linearly onto V_h. The optimal approximation u_h satisfies the Galerkin orthogonality $a(u_h - u, v) = 0$ for all $v \in V_h$. In this equation, a is a bounded, symmetric bilinear form that is coercive on V and depends on the chosen finite element model [1].

By contrast, in the Logarithmic finite element method, the space of degrees of freedom maps onto a generally non-linear submanifold M_h of the exact solution space V. As a result, the Galerkin orthogonality must be replaced by a necessary, though not sufficient, condition for an optimal approximation, $a(u_h - u, v) = 0$ for all $v \in T_{u_h} M_h$. A critical point satisfying this condition may not be a global, or even a local, minimum with regard to the distance of u_h and u in V. Note, however, that the Ritz-Galerkin method is often applied to a linearization of the global optimization problem, which is generally non-linear. In this case, an optimal approximation on V_h that satisfies the Galerkin orthogonality generally is not an optimal approximation to the solution of the global optimization problem.

If the Lie group G in equation (4.1a) and its group action on $\mathbf{x}_0$ are translational, then the manifold M_h is a linear subspace of V, and the observations with regard to the approximation in the Ritz-Galerkin method apply, as $T_{u_h} M_h = M_h$ if $M_h \leq V$.

6 A LogFE beam formulation

In this section, we illustrate how the Logarithmic finite element method can be used to formulate the kinematics of a prismatic beam in n dimensions, $n \in \{2,3\}$.

6.1 Transformations of Bernoulli and Timoshenko beam elements

The initial configuration of a beam element $\tilde{\kappa}_0^e$ is given as an element of the discretized constraint manifold of the element, $\tilde{\mathcal{C}}_h^e$. It consists of the positions $\mathbf{x}_0(\boldsymbol{\xi})$ of the centerline of the beam, possibly endowed with the orientations of the cross-section at each point of the centerline. The orientation is given as an element $\boldsymbol{\theta}_0(\boldsymbol{\xi})$ of the special orthogonal group $SO(n,\mathbb{R})$. For Bernoulli kinematics, only the positions of the centerline and their derivatives are being considered. For Timoshenko kinematics, a transformation $\chi^e(\boldsymbol{\xi})$ (see section 4.1) of $\boldsymbol{\theta}_0(\boldsymbol{\xi})$ may need to include a projection onto $SO(n,\mathbb{R})$, possibly as part of the group action σ_G of G on $\tilde{\kappa}_0^e$, in order to ensure that the resulting orientation $\boldsymbol{\theta}_\tau(\boldsymbol{\xi})$ lies in $SO(n,\mathbb{R})$.

A salient feature of the Logarithmic finite element method, as applied to beam models, is that the rotational degrees of freedom impact on the translational field values, i.e. the displacement of the centerline of the beam. Translational degrees of freedom do not impact on the rotational field values, i.e. the orientation of the cross-sections along the centerline. Careful construction of the Lie algebra $\mathfrak{g}$, as well as of the scalar shape functions $N(\boldsymbol{\xi})$, is essential in order to restrict the interaction of the rotational and the translational component of a transformation to the interior of the finite elements, while ensuring the separation of the impact of both types of degrees of freedom at the nodes, i.e. at the border of the finite elements [12].

6.2 Constructing the Lie algebra

The Lie algebra $\mathfrak{g}$ that we will use to construct the transformation function χ^e of a beam model based on the Logarithmic finite element method is isomorphic to a subgroup of the Lie algebra $\mathfrak{gl}(3n+1,\mathbb{R}) = \mathfrak{L}\big(GL(3n+1,\mathbb{R})\big)$. However, if we would directly use the multiplicative operation of the general linear group, we would end up with calculations involving $(3n+1)^2$ scalar elements for each interpolation point of the finite element model. Thus, we will construct the Lie algebra $\mathfrak{g}$ as the composition of direct and semidirect products of smaller and well-known Lie algebras (see section 3.2).

The basic rotational component of the Lie algebra $\mathfrak{g}$ is given by $\mathfrak{r}(n) = \mathfrak{L}\big(R(n)\big) := \mathfrak{so}(n,\mathbb{R}) \oplus \mathfrak{gl}(1,\mathbb{R})$. $R(n) := SO(n,\mathbb{R}) \otimes GL(1,\mathbb{R})^+ = \big\langle \exp \mathfrak{r}(n) \big\rangle$. [4] This component of $\mathfrak{g}$ governs rotations and dilatations of the centerline and rotations of the cross-sections. For $n=2$, $\mathfrak{r}(n) \cong \mathfrak{gl}(1,\mathbb{C}) = \mathfrak{L}\big(GL(1,\mathbb{C})\big)$. The embedding of $\mathfrak{r}(3)$ into $\mathfrak{gl}(3,\mathbb{R})$ is given by

$$\iota_1 : \mathfrak{r}(n) \hookrightarrow \mathfrak{gl}(n,\mathbb{R}),\ \big((r_1,r_2,r_3),s\big) \mapsto \begin{pmatrix} s & -r_3 & r_2 \\ r_3 & s & -r_1 \\ -r_2 & r_1 & s \end{pmatrix} \tag{6.1}$$

The Lie bracket on $\mathfrak{r}(3)$, the exponential function $\exp_{R(3)}$, and the product operation on the associated Lie group $R(3)$ can be derived from the product operation on $GL(3,\mathbb{R})$. The group action of $R(3)$ on sets such as $\mathbb{R}^3$ and $M_3(\mathbb{R})$ can be derived from the respective group action of $GL(3,\mathbb{R})$.

The basic translational element of $\mathfrak{g}$ is given by the translational Lie algebra $\mathfrak{t}(n,\mathbb{R})$, which is isomorphic to $\mathbb{R}^n$ as an additive group.

We now construct the component of $\mathfrak{g}$ that governs the rotations and translations related to one node of beam element. The Lie bracket of elements of the standard semidirect product $\mathfrak{t}(n) \rtimes_\beta \mathfrak{r}(n)$, $\beta(b,t) = \tau_b(t) = bt$, with bt given by the matrix multiplication, would not generally vanish even if the Lie bracket of their components in $\mathfrak{r}(n)$ vanishes. We therefore construct a direct product $\mathfrak{s}(n) := \mathfrak{t}(n) \oplus \mathfrak{r}(n)$, which, as a subalgebra of $\mathfrak{gl}(2n+1,\mathbb{R})$, is given by the embedding

$$\iota_2 : \mathfrak{t}(n) \oplus \mathfrak{r}(n) \hookrightarrow \mathfrak{gl}(2n+1,\mathbb{R}),\ (t,r) \mapsto \begin{pmatrix} r & -r & t \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.2}$$

The Lie algebra $\mathfrak{s}(n)$ is closed under the Lie bracket, i.e. $\big[\mathfrak{s}(n), \mathfrak{s}(n)\big] \subseteq \mathfrak{s}(n)$. The Lie bracket of this subgroup vanishes if, and only if, the Lie bracket of the canonical projec-

[4] $GL(1,\mathbb{R})^+$ consists of the 1×1 real matrices with positive determinant and is isomorphic to $\mathbb{R}^+$.

tions of the elements onto $\mathfrak{r}(t)$ vanishes:

$$\left[\begin{pmatrix} r & -r & t \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} r' & -r' & t' \\ 0 & 0 & t' \\ 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} [r,r'] & -[r,r'] & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \iota_2\Big(\big(0,[r,r']\big)\Big).^{5} \tag{6.3}$$

Finally, in order to combine the deformations related to the respective nodes of the beam element, we construct the Lie algebra $\mathfrak{g} := \mathfrak{q}(n)$ as a subalgebra of $\mathfrak{gl}(3n+1,\mathbb{R})$ based on the embedding

$$\iota_3 : \mathfrak{s}(n) \times \mathfrak{s}(n) \hookrightarrow \mathfrak{gl}(3n+1,\mathbb{R}), \big((t_1,r_1),(t_2,r_2)\big) \mapsto \begin{pmatrix} \bar{r} & -r_1 & -r_2 & \bar{t} \\ 0 & 0 & 0 & t_1 \\ 0 & 0 & 0 & t_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.4}$$

with $\bar{r} := r_1 + r_2$, $\bar{t} := t_1 + t_2$. The Lie bracket on the subspace resulting from the embedding ι_3, inherited from $\mathfrak{gl}(3n+1,\mathbb{R})$, is given as

$$\begin{aligned} &\left[\begin{pmatrix} \bar{r} & -r_1 & -r_2 & \bar{t} \\ 0 & 0 & 0 & t_1 \\ 0 & 0 & 0 & t_2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \bar{r}' & -r_1' & -r_2' & \bar{t}' \\ 0 & 0 & 0 & t_1' \\ 0 & 0 & 0 & t_2' \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] = \\ &\quad = \begin{pmatrix} [\bar{r},\bar{r}'] & -\bar{r}r_1' + \bar{r}'r_1 & -\bar{r}r_2' + \bar{r}'r_2 & \bar{r}\bar{t}' - \bar{r}'\bar{t} + \sum_{k=1}^{2}\left(-r_k t_k' + r_k' t_k\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{6.5}$$

The Lie bracket of elements of this subspace does not identically vanish, and the subspace is *not* closed under the Lie bracket. However, each subspace $V_{\mathfrak{s}(n)}$ of the Lie algebra $\mathfrak{q}(n)$, when endowed with the Lie bracket on $\mathfrak{q}(n)$, is a subalgebra of $\mathfrak{q}(n)$, i.e. it is closed under the Lie bracket on $\mathfrak{q}(n)$.

The non-commutativity of elements of $\mathfrak{q}(n)$ outside of the subalgebras $\mathfrak{s}(n)$ induces a strong coupling between the rotational and the translational component of the transformation. At the nodes, given appropriate shape functions, $X(\boldsymbol{\xi})$, and its derivatives with regard to $\boldsymbol{\xi}$, up to the required order, lie in $\mathfrak{s}(n)$, ensuring the independence of rotations and translations with regard to those nodal degrees of freedom of the beam element that may be associated with global degrees of freedom of the finite element system.

[5] For the standard semidirect product, the Lie bracket, as an operation on $\mathfrak{gl}(n+1,\mathbb{R})$, would read as

$$\left[\begin{pmatrix} r & t \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} r' & t' \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} [r,r'] & rt' - r't \\ 0 & 0 \end{pmatrix}.$$

6.3 The Lie group and its action on the initial configuration

The exponential function, on the subspace of $\mathfrak{gl}(3n+1,\mathbb{R})$, is given by

$$\exp\begin{pmatrix}\bar{r} & -r_1 & -r_2 & \bar{t}\\ 0 & 0 & 0 & t_1\\ 0 & 0 & 0 & t_2\\ 0 & 0 & 0 & 0\end{pmatrix}=\begin{pmatrix}\mathrm{e}^{\bar{r}} & -\zeta_1(\bar{r})r_1 & -\zeta_1(\bar{r})r_2 & \zeta_1(\bar{r})\bar{t}-\zeta_2(\bar{r})\sum\limits_{k=1}^{2}r_k t_k\\ 0 & I_n & 0 & t_1\\ 0 & 0 & I_n & t_2\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{6.6a}$$

with

$$\zeta_m : x \mapsto \sum_{k=0}^{\infty}\frac{1}{(k+m)!}x^k=\frac{\mathrm{e}^x-\sum_{k=0}^{m-1}\frac{1}{k!}x^k}{x^m}. \tag{6.6b}$$

Thus, $\mathrm{e}^{\bar{r}}\in R(n)=SO(n,\mathbb{R})\otimes GL(1,\mathbb{R})^+$ is the only power series that must be evaluated in order to calculate the exponential on $\mathfrak{q}(n)$. We recall that the isomorphism of the Lie algebra $\mathfrak{g}$ to a subalgebra of $\mathfrak{gl}(3n+1,\mathbb{R})$ ensures that operations on $\mathfrak{g}$ can be derived from the matrix multiplication on $M_{3n+1}(\mathbb{R})$, and that actual calculations can be performed using these operations, without involving multiplications on $M_{3n+1}(\mathbb{R})$.

Given the spatial positions of the nodes of the beam element in the initial configuration as $\mathbf{x}_0^1$ and $\mathbf{x}_0^2$, the group action on the initial configuration $\tilde{\kappa}_0$ is given by

$$\sigma_{g(\boldsymbol{\xi})}:\{\tilde{\kappa}_0\}\to\tilde{\mathcal{Q}},\ \big(\boldsymbol{\xi},(\mathbf{x}_0,\boldsymbol{\theta}_0)\big)\mapsto\left(\boldsymbol{\xi},\left(\pi_1\circ\big(\varphi\circ g(\boldsymbol{\xi})\big)\left(\mathbf{x}_0,\mathbf{x}_0^1,\mathbf{x}_0^2,1\right)^{\mathrm{T}},\pi_2\big(\mathrm{e}^{\bar{r}}\boldsymbol{\theta}_0\big)\right)\right),^{6} \tag{6.7}$$

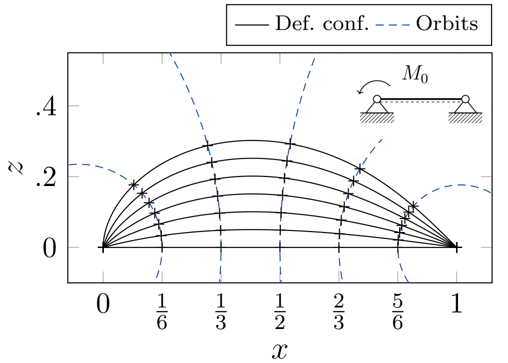
with $\varphi\circ g(\boldsymbol{\xi})=\varphi\circ\exp\big(X(\boldsymbol{\xi})\big)=\exp\big(\iota_3\circ X(\boldsymbol{\xi})\big)$. In the expression $\mathrm{e}^{\bar{r}}$, we assume that $X(\boldsymbol{\xi})=\big((t_1,r_1),(t_2,r_2)\big)\in\mathfrak{s}(n)\times\mathfrak{s}(n)$. The calculation of the element $X(\boldsymbol{\xi})$ of the Lie algebra $\mathfrak{g}$ is given in equation (4.1c). π_1 projects a vector in $\mathbb{R}^{3n+1}$ onto its n leading dimensions, π_2 projects an element of $R(n)$ onto $SO(n,\mathbb{R})$. Figure 1 shows the deformation of the initial configuration of a beam as a result of the group action of $\exp\big(sX(\boldsymbol{\xi})\big)$, as the value of $s\in\mathbb{R}$ increases linearly.

Results of the approximation of the deformation of a beam element endowed with Bernoulli kinematics for different Dirichlet and Neumann boundary conditions are presented in [12] and show good agreement with reference solutions based on the standard Ritz-Galerkin method. The results of a Timoshenko beam formulation, while satisfactory for coarse discretizations, indicate the need for a co-rotational formulation in order to ensure convergence of the results with mesh refinement [13].

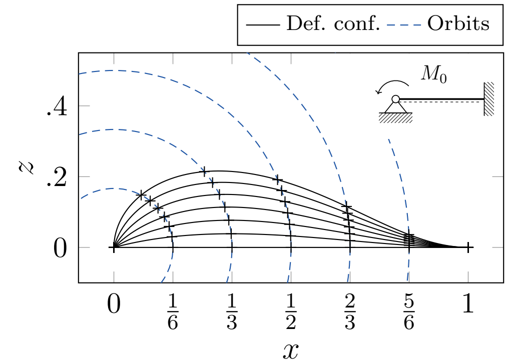
7 Conclusion

The Logarithmic finite element method, which extends the Ritz-Galerkin method from a linear subspace of the exact solution space to a non-linear manifold, provides a novel approach to the formulation of finite elements.

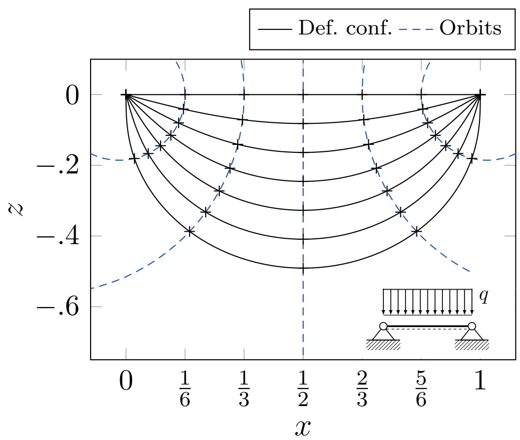
[6] For brevity, the description of the map refers to the single material points, rather than to the configuration as a whole.



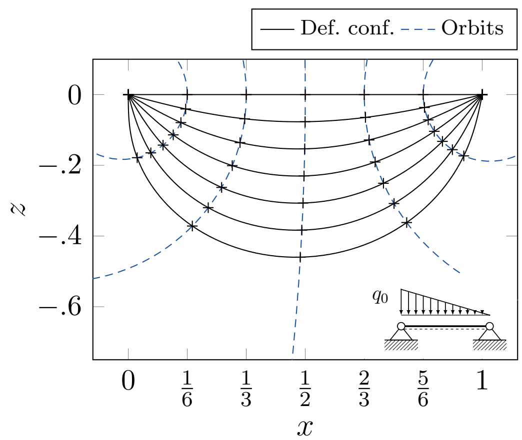
(a) Simply supported beam, external moment on left support



(b) Simply supported/clamped beam, external moment on left support



(c) Simply supported, constant line load



(d) Simply supported, linearly decreasing line load

Figure 1: Transformation of the initial configuration $\tilde{\kappa}_0 : [0,1] \to \mathbb{R}^2$, $\boldsymbol{\xi} \mapsto (\boldsymbol{\xi}, 0)$, of a beam by $\exp\big(sX(\boldsymbol{\xi})\big)$, $s \in \mathbb{R}$, $X(\boldsymbol{\xi}) \in \mathfrak{g}$. The figures depict the current configurations and the orbits $\mathbf{x}(\boldsymbol{\xi}, s)$ of points on the neutral axis of the beam, as the value of s increases linearly.

Beam elements can be formulated based on matrix Lie algebras, and the product operations on the Lie algebra and the associated Lie group can be derived from the matrix multiplication. However, careful construction of the required Lie algebra is essential in order to allow for a coupling of the components of the deformation in the interior of the element, as well as for a separation of the impact of the degrees of freedom at the nodes.

The Logarithmic finite element method offers a new approach to the construction of finite element formulations for various structural elements. While the flexibility provided by opening up the approximation space to a non-linear manifold of the exact solution space offers new possibilities, actual finite element formulations must preserve those characteristics of the standard Ritz-Galerkin method that enable the smooth integration of different finite elements into a global finite element system.

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