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SENSITIVITY ANALYSIS AS A TOOL FOR OPTIMAL MATERIAL DESIGN

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Abstract. The presented contribution discusses the derivation of design sensitivity information of physical reaction forces arising in mechanical systems. The obtained gradient information can be used for the design of support areas within problems stated on single scales as well as for the design of microstructures in terms of representative volume elements (RVE) within numerical homogenisation techniques. Especially, FE^2 approaches based on Langrange multiplier methods, where the Lagrange multiplier itself can be connected to some force or traction values on the surface of the RVE, can benefit from introduced and presented relations.

1 Introduction

Structural optimisation has a long tradition and has been investigated for many years and in a variety of fields. Techniques for numerical homogenisation, i.e. FE^2 methods, allow investigations of the physical behaviour of complex heterogeneous materials and lead to a remarkable number of applications and real world problems, see [15, 17] and references therein for details. A combination of both established methods leads to a significant increase of possible fields of applications and justifies its eminent importance. Besides mathematical algorithms, sensitivity analysis is a fundamental topic within solution strategies for optimisation problems, especially within techniques based on gradient information. Its realisation is responsible for the efficiency and accuracy of used methods. In this context several works conclude, that performing sensitivity analysis using variational methods according to [11] or [8, 9] seems to be a particularly promising approach to design sensitivity analysis. Especially the *enhanced intrinsic formulation* proposed by [2] and [1, 3] provides many beneficial advantages.

In some cases, it is interesting to analyse support areas of given systems to get detailed information about the interaction between considered structural parts and the ground. The profile of distributed forces, tractions or stresses can be investigated in order to make predictions about mechanical behaviour or even about possible failure. Using some advanced information it is also possible to improve and optimise several kinds of support areas. For instance, methods for simultaneous design of structures and supports using techniques from topology optimisation were proposed in [6]. Here, supports have been introduced as a new subset of design variables within the optimisation process for minimum compliance and mechanism design with the target to find the optimal location of supports. A similar approach can be found in [25]. Aspects concerning optimal design of supports for beam and frame structures in general were reported in [5]. Several influences, e.g. number of supports or their position and stiffness, were investigated and, in a similar fashion, publication [4] demonstrates a method for the determination of the overall number, position and generalised forces of actuators in smart structures. The authors in [19] tackled the shape design of rectangular support blocks and foundations of machines in order to reduce mass. Their studies focused on external dynamic forces and loads coming from the soil. Introduced constraints are horizontal and vertical amplitudes of forces and stresses on the soil ground. The topics in [21] can be put in a similar context. The authors presented methods for optimisation of boundary conditions subjected to maximum fundamental frequency of vibrating structures in order to find optimal locations. A variational formulation and the approach of material derivatives was the foundation in their gradient-based optimisation techniques. Another field of application was discussed in [7], where an algorithm was proposed for shape optimisation of contact problems with desired contact traction distribution on specified contact surfaces or areas. It should be mentioned, that the quantity of interest was the distribution of forces or tractions and therefore, it can be related to the presented sensitivity analysis of reaction forces. In contrast to the topic of this contribution, the influence of the position of externally applied constant loads or forces on structural response, such as nodal displacements, mean compliance and stress, can be investigated and was done in [22]. The common aspect is, that in both cases sensitivity of forces plays the central role, on one hand on the active side (applied force) and on the other hand on the passive side (reaction force). The authors in [24] investigated shape design sensitivities with respect to kinematical boundaries, i.e. influence on structural response due to modifications on the Dirichlet boundary Γ_{D}. Although the contribution at hand has a different topic and intention, the support area is the domain of interest in both cases, i.e. in terms of kinematical boundaries and sensitivity analysis of reaction forces.

The purpose of the presented study is to describe and examine a sensitivity relation for reaction forces based on available tangent operators and the sensitivity relation for the state. The obtained gradient information can be used to set up optimisation problems and to find optimal designs with respect to the distribution of reaction forces. The overall amplitude of maximum reaction forces can be controlled and adjusted in combination with several objective functionals, e.g. compliance or volume. The presented approach is not comparable to previously referred works directly, but it can be seen as an extension to the variety of available methods for analysis and design of support areas. Further advantage is that derived relations can be transferred to optimisation problems on multiple scales, where effective quantities of RVEs on the microscale are formulated in terms of tractions or forces on the boundary, see [14, 15, 17] on theoretical and numerical aspects and explanations on multiscale methods.

2 Design sensitivity analysis

In solid mechanics and other fields of computational mechanics, especially in the finite element framework, the *weak form of equilibrium* $R(\boldsymbol{v}, \boldsymbol{s}; \boldsymbol{\eta}) = 0$ plays a central role. It depends on a general state variable $\boldsymbol{v} \in \mathcal{V}$, a design variable $\boldsymbol{s} \in \mathcal{S}$, and any test function $\boldsymbol{\eta} \in \mathcal{V}$. The weak form is of similar importance for the setting of design sensitivity analysis within structural optimisation and is often incorporated as an equality constraint within the posed optimisation problem. It has to be fulfilled for any arbitrary state and design. Its variation leads to the well-known *tangent stiffness operator* for structural analysis and also to the *tangent pseudo load operator*, which is used to formulate sensitivity relations to describe effects in the physical space due to modifications in the material space. Altogether, derivatives or variations, especially partial variations of objective functionals and constraints with respect to the state and design, are essential for gradient-based solution strategies. Fundamental relations for the optimisation of problems from structural mechanics are provided in this section. The variation of physical reaction forces as objective or constraint functional is presented in Section 3 and investigated numerically in Section 4.

Remark 2.1 (Notation) *In general, the total variation of a quantity* $(\cdot)(\boldsymbol{v}, \boldsymbol{s})$ *is given by the partial variation with respect to* $\boldsymbol{v}$ *and a fixed design* $\hat{\boldsymbol{s}}$ *as well as the partial variation with respect to* $\boldsymbol{s}$ *and a fixed state* $\hat{\boldsymbol{v}}$*. For a more compact representation, the notation* $\delta(\cdot) = (\cdot)'$ *is used and total and partial variations are formulated in the following compact way*

$$\delta(\cdot)(\boldsymbol{v}, \boldsymbol{s}) = (\cdot)'_v(\boldsymbol{v}, \hat{\boldsymbol{s}}) + (\cdot)'_s(\hat{\boldsymbol{v}}, \boldsymbol{s}) = (\cdot)'(\boldsymbol{v}, \boldsymbol{s}). \tag{1}$$

Without going into detail, the compilation of all following variations is based on the publications [1] and [13]. The variation of the weak form can be motivated in several ways, e.g. minimisation principle of overall energy, solution of an optimisation problem for an arbitrary objective function with an equilibrium constraint, or finally the simple statement, that any perturbation in the design space must not violate the physical equilibrium state. Therefore, the total variation of the nonlinear residual can be investigated

$$R' = R'_v(\boldsymbol{v}, \boldsymbol{s}; \boldsymbol{\eta}, \delta\boldsymbol{v}) + R'_s(\boldsymbol{v}, \boldsymbol{s}; \boldsymbol{\eta}, \delta\boldsymbol{s}) = k(\boldsymbol{v}, \boldsymbol{s}; \boldsymbol{\eta}, \delta\boldsymbol{v}) + p(\boldsymbol{v}, \boldsymbol{s}; \boldsymbol{\eta}, \delta\boldsymbol{s}) = 0. \tag{2}$$

The variations of the physical residual with respect to $\boldsymbol{v}$ and $\boldsymbol{s}$ are introduced by the tangent operators $k(\boldsymbol{v}, \boldsymbol{s}; \boldsymbol{\eta}, \delta\boldsymbol{v}) = R'_v(\boldsymbol{v}, \boldsymbol{s}; \boldsymbol{\eta}, \delta\boldsymbol{v})$ for physical stiffness and $p(\boldsymbol{v}, \boldsymbol{s}; \boldsymbol{\eta}, \delta\boldsymbol{s}) = R'_s(\boldsymbol{v}, \boldsymbol{s}; \boldsymbol{\eta}, \delta\boldsymbol{s})$ for pseudo load. The solution of the sensitivity relation from Eq. (2) allows the derivation of the *implicit sensitivity of the state* in current equilibrium point $(\boldsymbol{v}, \boldsymbol{s})$

$$\delta\boldsymbol{v} = s(\boldsymbol{v}, \boldsymbol{s}; \delta\boldsymbol{s}). \tag{3}$$

After standard finite element discretisation with the discrete approximation $\boldsymbol{v}_h$ for the state and $\boldsymbol{s}_h$ for design, the discrete parameters $\mathsf{v} \in \mathbb{R}^{n_v}$ and $\mathsf{s} \in \mathbb{R}^{n_s}$ can be used to obtain the matrix description of the continuous forms, cf. [23] for instance. The approximations for the corresponding variations $\delta\mathsf{v} \in \mathbb{R}^{n_v}$ and $\delta\mathsf{s} \in \mathbb{R}^{n_s}$ as well as for the test function $\boldsymbol{\eta} \in \mathbb{R}^{n_v}$ are chosen in the same manner. The overall number of the discrete state variables in $\mathcal{V}_h \subset \mathcal{V}$ is given by n_v, and n_s is the number of discrete design parameters in $\mathcal{S}_h \subset \mathcal{S}$.

$$R(\boldsymbol{v}_h, \boldsymbol{s}_h; \boldsymbol{\eta}_h) = \boldsymbol{\eta}^T \mathsf{R}, \qquad \mathsf{R} \in \mathbb{R}^{n_v}, \tag{4}$$

$$k(\boldsymbol{v}_h, \boldsymbol{s}_h; \boldsymbol{\eta}_h, \delta\boldsymbol{v}_h) = \boldsymbol{\eta}^T \mathsf{K} \delta\mathsf{v}, \qquad \mathsf{K} \in \mathbb{R}^{n_v \times n_v}, \tag{5}$$

$$p(\boldsymbol{v}_h, \boldsymbol{s}_h; \boldsymbol{\eta}_h, \delta\boldsymbol{s}_h) = \boldsymbol{\eta}^T \mathsf{P} \delta\mathsf{s}, \qquad \mathsf{P} \in \mathbb{R}^{n_v \times n_s}. \tag{6}$$

The discrete form of the variation of the weak form in Eq. (2) evaluated in $(\boldsymbol{v}_h, \boldsymbol{s}_h)$ reads Eq. $(7)_1$ and thus, due to the arbitrariness of the test function $\boldsymbol{\eta}$, Eq. $(7)_2$ holds true.

$$R' = \boldsymbol{\eta}^T \boldsymbol{R}' = \boldsymbol{\eta}^T \left[\boldsymbol{K}\delta\boldsymbol{v} + \boldsymbol{P}\delta\boldsymbol{s}\right] = 0, \qquad \boldsymbol{R}' = \boldsymbol{K}\delta\boldsymbol{v} + \boldsymbol{P}\delta\boldsymbol{s} = \boldsymbol{0}. \tag{7}$$

Eq. $(7)_2$ is utilised for the evaluation of the discrete form of the sensitivity relation from Eq. (3)

$$\delta\boldsymbol{v} = \boldsymbol{S}\delta\boldsymbol{s} \quad \text{with} \quad \boldsymbol{S} = -\boldsymbol{K}^{-1}\boldsymbol{P} \quad \text{and} \quad \boldsymbol{S} \in \mathbb{R}^{n_v \times n_s} \tag{8}$$

for any arbitrary design variation $\delta\boldsymbol{s}$. The introduced matrix $\boldsymbol{S}$ is the so called *design sensitivity matrix*. It connects variations in the material space with variations in the physical space and allows predictions of changes in the state $\boldsymbol{v}$ due to design modifications $\delta\boldsymbol{s}$.

Remark 2.2 (Choice of design parameters) *Using finite element techniques, a general structural analysis of given systems is based on discrete nodal finite element coordinates $\boldsymbol{X}$. Within so-called parameter free optimisation, sensitivity analysis is performed with respect to $\boldsymbol{X}$. If the geometry description is realised using computer aided geometric design (CAGD), coordinates of control points of Bézier curves can be chosen as design variables. The design parameters $\boldsymbol{X}$ depend then on the new defined design variables $\boldsymbol{s}$, i.e. $\boldsymbol{X}(\boldsymbol{s})$. Therefore, the sensitivities with respect to $\boldsymbol{X}$ have to be transformed via a design velocity fields matrix $\boldsymbol{V} = \mathrm{d}\boldsymbol{X}/\mathrm{d}\boldsymbol{s}$ into the chosen design space. The final discrete variation of an arbitrary functional $f(\boldsymbol{v}, \boldsymbol{X}(\boldsymbol{s}))$ reads*

$$f' = \frac{\partial f}{\partial \boldsymbol{v}}\delta\boldsymbol{v} + \frac{\partial f}{\partial \boldsymbol{X}}\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\boldsymbol{s}}\delta\boldsymbol{s} = \left(\frac{\partial f}{\partial \boldsymbol{v}}\,\boldsymbol{S} + \frac{\partial f}{\partial \boldsymbol{X}}\right)\boldsymbol{V}\delta\boldsymbol{s}, \tag{9}$$

with the matrix $\boldsymbol{S}$ being the total derivative of the state $\boldsymbol{v}$ with respect to nodal coordinates $\boldsymbol{X}$, i.e. $\boldsymbol{S} = \mathrm{d}\boldsymbol{v}/\mathrm{d}\boldsymbol{X}$. For further hints and explanations on design velocity fields see [8, 9].

The general discrete optimisation problem with an objective functional J, various constraints $\boldsymbol{h}, \boldsymbol{g}$, lower and upper side constraints $\boldsymbol{s}^{\mathrm{l}}, \boldsymbol{s}^{\mathrm{u}}$ can be introduced in the following abstract way.

Problem 1 (General discrete optimisation problem) *Find $\{\boldsymbol{v}_h, \boldsymbol{s}_h\} \in \mathcal{V}_h \times \mathcal{S}_h$ of the discrete objective functional $J : \mathcal{V}_h \times \mathcal{S}_h \to \mathbb{R}$ such that*

$$\min_{\boldsymbol{v},\boldsymbol{s} \in \mathcal{V}_h \times \mathcal{S}_h} J(\boldsymbol{v}, \boldsymbol{s}) \quad \textit{subject to the constraints} \quad \boldsymbol{h}(\boldsymbol{v}, \boldsymbol{s}) = \boldsymbol{0}, \quad \boldsymbol{g}(\boldsymbol{v}, \boldsymbol{s}) \le \boldsymbol{0}, \quad \boldsymbol{s}^{\mathrm{l}} \le \boldsymbol{s} \le \boldsymbol{s}^{\mathrm{u}} \tag{10}$$

with $\boldsymbol{h}(\boldsymbol{v}, \boldsymbol{s})$ and $\boldsymbol{g}(\boldsymbol{v}, \boldsymbol{s})$ being matrix representations of equality and inequality constraints.

3 Sensitivity analysis of physical reaction forces

For the derivation of the sensitivity relation for the physical reaction forces only parts of the discrete physical residual in Eq. (4) have to be considered. Therefore, some arrangements are necessary in advance. The discrete relations presented in Section 2 have to be partitioned in internal and boundary parameters as specified in the following. With regard to the discrete formulation of derived sensitivity relations, the discrete parameters, i.e. the state parameters $\boldsymbol{v}$ and design parameters $\boldsymbol{X}$, are partitioned into contributions on the Dirichlet boundary Γ_{D}, the

Neumann boundary Γ_{N} and into those of the inner domain Ω. With the overall representation of the domain $\overline{\Omega} = \Omega \cup \Gamma$ and the boundary $\Gamma = \Gamma_{\mathrm{N}} \cup \Gamma_{\mathrm{D}}$, where $\Gamma_{\mathrm{N}} \cap \Gamma_{\mathrm{D}} = \emptyset$, all appearing quantities can be identified by the notation

$$\begin{aligned} \text{State:} \quad & (\cdot)_{\mathrm{a}} \in \Omega \cup \Gamma_{\mathrm{N}} \quad && \text{and} \quad && (\cdot)_{\mathrm{b}} \in \Gamma_{\mathrm{D}}, \\ \text{Design:} \quad & (\cdot)_{\mathrm{A}} \in \Omega \cup \Gamma_{\mathrm{N}} \quad && \text{and} \quad && (\cdot)_{\mathrm{B}} \in \Gamma_{\mathrm{D}} \end{aligned} \tag{11}$$

with the set (a, b) for the partition of state variables and the set (A, B) for the partition of design parameters. The number of state quantities $(\cdot)_{\mathrm{a}}$ is given by n_{a}, and the number of state quantities $(\cdot)_{\mathrm{b}}$ is given by n_{b}. Same holds true for the design partition and therefore, the number of design quantities $(\cdot)_{\mathrm{A}}$ is given by n_{A}, and the number of design quantities $(\cdot)_{\mathrm{B}}$ is given by n_{B}. It is useful to introduce these different kinds of subsets because it finally allows to define and prescribe different types of boundary conditions, i.e. explicit boundary conditions in physical and explicit boundary conditions in material space. Using this definition, the state $\boldsymbol{v}$, the variation of the state $\delta\boldsymbol{v}$, the design parameters $\boldsymbol{X}$ and the variation of the design parameters $\delta\boldsymbol{X}$ are divided

$$\boldsymbol{v} = \begin{bmatrix} \boldsymbol{v}_{\mathrm{a}} & \boldsymbol{v}_{\mathrm{b}} \end{bmatrix}^T, \quad \delta\boldsymbol{v} = \begin{bmatrix} \delta\boldsymbol{v}_{\mathrm{a}} & \delta\boldsymbol{v}_{\mathrm{b}} \end{bmatrix}^T, \quad \boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_{\mathrm{A}} & \boldsymbol{X}_{\mathrm{B}} \end{bmatrix}^T, \quad \delta\boldsymbol{X} = \begin{bmatrix} \delta\boldsymbol{X}_{\mathrm{A}} & \delta\boldsymbol{X}_{\mathrm{B}} \end{bmatrix}^T \tag{12}$$

with dimensions $\{\boldsymbol{v}_{\mathrm{a}}, \delta\boldsymbol{v}_{\mathrm{a}}\} \in \mathbb{R}^{n_{\mathrm{a}}}$, $\{\boldsymbol{v}_{\mathrm{b}}, \delta\boldsymbol{v}_{\mathrm{b}}\} \in \mathbb{R}^{n_{\mathrm{b}}}$, $\{\boldsymbol{X}_{\mathrm{A}}, \delta\boldsymbol{X}_{\mathrm{A}}\} \in \mathbb{R}^{n_{\mathrm{A}}}$ and $\{\boldsymbol{X}_{\mathrm{B}}, \delta\boldsymbol{X}_{\mathrm{B}}\} \in \mathbb{R}^{n_{\mathrm{B}}}$. As a consequence, a similar partition holds true for the physical residual vector in Eq. (4) with the dimensions $\boldsymbol{R}_{\mathrm{a}} \in \mathbb{R}^{n_{\mathrm{a}}}$, $\boldsymbol{R}_{\mathrm{b}} \in \mathbb{R}^{n_{\mathrm{b}}}$, as well as for the variation of the residual vector Eq. $(7)_2$

$$\boldsymbol{R}(\boldsymbol{v}, \boldsymbol{X}; \boldsymbol{\eta}) = \begin{bmatrix} \boldsymbol{R}_{\mathrm{a}}(\boldsymbol{v}_{\mathrm{a}}, \boldsymbol{v}_{\mathrm{b}}, \boldsymbol{X}_{\mathrm{A}}, \boldsymbol{X}_{\mathrm{B}}; \boldsymbol{\eta}) \\ \boldsymbol{R}_{\mathrm{b}}(\boldsymbol{v}_{\mathrm{a}}, \boldsymbol{v}_{\mathrm{b}}, \boldsymbol{X}_{\mathrm{A}}, \boldsymbol{X}_{\mathrm{B}}; \boldsymbol{\eta}) \end{bmatrix}, \qquad \boldsymbol{R}' = \begin{bmatrix} \boldsymbol{R}'_{\mathrm{a}} \\ \boldsymbol{R}'_{\mathrm{b}} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{R}_{\mathrm{a}})'_v + (\boldsymbol{R}_{\mathrm{a}})'_X \\ (\boldsymbol{R}_{\mathrm{b}})'_v + (\boldsymbol{R}_{\mathrm{b}})'_X \end{bmatrix}. \tag{13}$$

Here, the partial variations of the partitioned residual $\boldsymbol{R}_{\mathrm{a}}, \boldsymbol{R}_{\mathrm{b}}$ have to be determined with respect to the partitioned state $\boldsymbol{v}_{\mathrm{a}}, \boldsymbol{v}_{\mathrm{b}}$ and with respect to the partitioned design $\boldsymbol{X}_{\mathrm{A}}, \boldsymbol{X}_{\mathrm{B}}$. The explicit partitioned representation of Eq. $(7)_2$ is of the form

$$\boldsymbol{R}' = \boldsymbol{R}'_v + \boldsymbol{R}'_X = \begin{bmatrix} (\boldsymbol{R}_{\mathrm{a}})'_v \\ (\boldsymbol{R}_{\mathrm{b}})'_v \end{bmatrix} + \begin{bmatrix} (\boldsymbol{R}_{\mathrm{a}})'_X \\ (\boldsymbol{R}_{\mathrm{b}})'_X \end{bmatrix} = \begin{bmatrix} \boldsymbol{K}_{\mathrm{aa}} & \boldsymbol{K}_{\mathrm{ab}} \\ \boldsymbol{K}_{\mathrm{ba}} & \boldsymbol{K}_{\mathrm{bb}} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{v}_{\mathrm{a}} \\ \delta\boldsymbol{v}_{\mathrm{b}} \end{bmatrix} + \begin{bmatrix} \boldsymbol{P}_{\mathrm{aA}} & \boldsymbol{P}_{\mathrm{aB}} \\ \boldsymbol{P}_{\mathrm{bA}} & \boldsymbol{P}_{\mathrm{bB}} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{X}_{\mathrm{A}} \\ \delta\boldsymbol{X}_{\mathrm{B}} \end{bmatrix} = \boldsymbol{0}, \tag{14}$$

with nodal coordinates $\boldsymbol{X}$ of the finite element nodes as design parameters. Resulting dimensions of obtained sub-matrices for the stiffness and pseudo load matrix can be specified by

$$\begin{aligned} &\boldsymbol{K}_{\mathrm{aa}} \in \mathbb{R}^{n_{\mathrm{a}} \times n_{\mathrm{a}}}, \ \boldsymbol{K}_{\mathrm{ab}} \in \mathbb{R}^{n_{\mathrm{a}} \times n_{\mathrm{b}}}, \ \boldsymbol{K}_{\mathrm{ba}} \in \mathbb{R}^{n_{\mathrm{b}} \times n_{\mathrm{a}}}, \ \boldsymbol{K}_{\mathrm{bb}} \in \mathbb{R}^{n_{\mathrm{b}} \times n_{\mathrm{b}}}, \\ &\boldsymbol{P}_{\mathrm{aA}} \in \mathbb{R}^{n_{\mathrm{a}} \times n_{\mathrm{A}}}, \ \boldsymbol{P}_{\mathrm{aB}} \in \mathbb{R}^{n_{\mathrm{a}} \times n_{\mathrm{B}}}, \ \boldsymbol{P}_{\mathrm{bA}} \in \mathbb{R}^{n_{\mathrm{b}} \times n_{\mathrm{A}}}, \ \boldsymbol{P}_{\mathrm{bB}} \in \mathbb{R}^{n_{\mathrm{b}} \times n_{\mathrm{B}}}. \end{aligned} \tag{15}$$

Relation (14) can be rearranged and allows the explicit computation of the sensitivity of the state from Eq. (8)

$$\delta\boldsymbol{v}_{\mathrm{a}} = -\boldsymbol{K}_{\mathrm{aa}}^{-1} \begin{bmatrix} \boldsymbol{P}_{\mathrm{aA}} & \boldsymbol{P}_{\mathrm{aB}} \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{X}_{\mathrm{A}} \\ \delta\boldsymbol{X}_{\mathrm{B}} \end{bmatrix} = -\boldsymbol{K}_{\mathrm{aa}}^{-1} \boldsymbol{P}_{\mathrm{a}} \delta\boldsymbol{X} = \boldsymbol{S}_{\mathrm{a}} \delta\boldsymbol{X}. \tag{16}$$

With $\boldsymbol{P}_{\mathrm{a}} = \begin{bmatrix} \boldsymbol{P}_{\mathrm{aA}} & \boldsymbol{P}_{\mathrm{aB}} \end{bmatrix}$, the resulting quantity $\boldsymbol{S}_{\mathrm{a}} = -\boldsymbol{K}_{\mathrm{aa}}^{-1} \boldsymbol{P}_{\mathrm{a}}$ contains the sensitivity information of the state in the inner domain Ω and on the Neumann boundary Γ_{N}. Compared to Eq. (8) it can be termed *reduced sensitivity matrix*. The part $(\cdot)_{\mathrm{b}}$ in the overall sensitivity matrix $\boldsymbol{S}$, i.e. $\boldsymbol{S}_{\mathrm{b}}$, vanishes due to the fact, that the boundary conditions for the displacements $\boldsymbol{v}_{\mathrm{b}}$ on the Dirichlet boundary Γ_{D} are fulfilled strongly and therefore, their variations $\delta\boldsymbol{v}_{\mathrm{b}}$ vanish.

3.1 Variation of the external part of physical residual

In structural analysis, equilibrium is fulfilled for a state variable $\boldsymbol{v}$ and a fixed design $\boldsymbol{X}$ if the residual equation $R(\boldsymbol{v}, \boldsymbol{s}; \boldsymbol{\eta}) = 0$ vanishes. Referring the partitioned residual $(13)_1$ and the split in *internal* and *external* contributions $\boldsymbol{R}^{\text{int}}(\boldsymbol{v}, \boldsymbol{X}; \boldsymbol{\eta})$ and $\boldsymbol{R}^{\text{ext}}(\boldsymbol{v}, \boldsymbol{X}; \boldsymbol{\eta})$ one obtains

$$\boldsymbol{R}(\boldsymbol{v}, \boldsymbol{X}; \boldsymbol{\eta}) = \boldsymbol{R}^{\text{int}}(\boldsymbol{v}, \boldsymbol{X}; \boldsymbol{\eta}) - \boldsymbol{R}^{\text{ext}}(\boldsymbol{v}, \boldsymbol{X}; \boldsymbol{\eta}) = \begin{bmatrix} \boldsymbol{R}_{\text{a}}^{\text{int}} \\ \boldsymbol{R}_{\text{b}}^{\text{int}} \end{bmatrix} - \begin{bmatrix} \boldsymbol{R}_{\text{a}}^{\text{ext}} \\ \boldsymbol{R}_{\text{b}}^{\text{ext}} \end{bmatrix} = \mathbf{0}. \tag{17}$$

Dimensions of the residual in Eq. (17) correspond to discussed quantities $\left\{\boldsymbol{R}, \boldsymbol{R}^{\text{int}}, \boldsymbol{R}^{\text{ext}}\right\} \in \mathbb{R}^{(n_{\text{a}}+n_{\text{b}})=n_v}$ and therefore $\left\{\boldsymbol{R}_{\text{a}}^{\text{int}}, \boldsymbol{R}_{\text{a}}^{\text{ext}}\right\} \in \mathbb{R}^{n_{\text{a}}}$, $\left\{\boldsymbol{R}_{\text{b}}^{\text{int}}, \boldsymbol{R}_{\text{b}}^{\text{ext}}\right\} \in \mathbb{R}^{n_{\text{b}}}$. In the solution $\boldsymbol{v}$, reaction forces on Γ_{D} of a given system are equal to their internal counterparts and can be computed using

$$\boldsymbol{R}_{\text{b}}^{\text{ext}}(\boldsymbol{v}, \boldsymbol{X}; \boldsymbol{\eta}) = \boldsymbol{R}_{\text{b}}^{\text{int}}(\boldsymbol{v}, \boldsymbol{X}; \boldsymbol{\eta}). \tag{18}$$

Remark, that relation (18) is only valid for fully converged solutions $\boldsymbol{v}$. Otherwise, errors are unavoidable and have a significant influence on following sensitivity relations. For the sensitivity relation of reaction forces, parts of already discussed sensitivity relations in Section 2 can be considered. The variation of an arbitrary function f is presented in Remark 2.2 and now, this principle can be transferred to the variation of reaction forces. Hence, variations with respect to the state and design are necessary

$$\left(\boldsymbol{R}_{\text{b}}^{\text{ext}}\right)' = \left(\boldsymbol{R}_{\text{b}}^{\text{int}}\right)' = \left(\boldsymbol{R}_{\text{b}}^{\text{int}}\right)'_v + \left(\boldsymbol{R}_{\text{b}}^{\text{int}}\right)'_X. \tag{19}$$

In contrast to the variation of the overall residual, only variations of the internal part $\boldsymbol{R}^{\text{int}}$ are necessary. For that reason, the investigation of the sensitivity relation for the residual $\boldsymbol{R}$ in terms of internal and external parts $\boldsymbol{R}^{\text{int}}$ and $\boldsymbol{R}^{\text{ext}}$ is conducted in the following. The total variation of the splitted residual in Eq. (17) reads

$$\boldsymbol{R}' = \left(\boldsymbol{R}^{\text{int}}(\boldsymbol{v}, \boldsymbol{X}; \boldsymbol{\eta}) - \boldsymbol{R}^{\text{ext}}(\boldsymbol{v}, \boldsymbol{X}; \boldsymbol{\eta})\right)' = \left(\boldsymbol{R}^{\text{int}}\right)'_v - \left(\boldsymbol{R}^{\text{ext}}\right)'_v + \left(\boldsymbol{R}^{\text{int}}\right)'_X - \left(\boldsymbol{R}^{\text{ext}}\right)'_X. \tag{20}$$

Using the resulting partial variation of $\boldsymbol{R}$ with respect to the state variable $\boldsymbol{v}$, cf. Eq. (12),

$$\boldsymbol{R}'_v = \begin{bmatrix} \left(\boldsymbol{R}_{\text{a}}^{\text{int}}\right)'_v - \left(\boldsymbol{R}_{\text{a}}^{\text{ext}}\right)'_v \\ \left(\boldsymbol{R}_{\text{b}}^{\text{int}}\right)'_v - \left(\boldsymbol{R}_{\text{b}}^{\text{ext}}\right)'_v \end{bmatrix} = \left\{ \begin{bmatrix} \boldsymbol{K}_{\text{aa}}^{\text{int}} & \boldsymbol{K}_{\text{ab}}^{\text{int}} \\ \boldsymbol{K}_{\text{ba}}^{\text{int}} & \boldsymbol{K}_{\text{bb}}^{\text{int}} \end{bmatrix} - \begin{bmatrix} \boldsymbol{K}_{\text{aa}}^{\text{ext}} & \boldsymbol{K}_{\text{ab}}^{\text{ext}} \\ \boldsymbol{K}_{\text{ba}}^{\text{ext}} & \boldsymbol{K}_{\text{bb}}^{\text{ext}} \end{bmatrix} \right\} \begin{bmatrix} \delta \boldsymbol{v}_{\text{a}} \\ \delta \boldsymbol{v}_{\text{b}} \end{bmatrix} \tag{21}$$

and the resulting partial variation of $\boldsymbol{R}$ with respect to design $\boldsymbol{X}$, cf. Eq. (12),

$$\boldsymbol{R}'_X = \begin{bmatrix} \left(\boldsymbol{R}_{\text{a}}^{\text{int}}\right)'_X - \left(\boldsymbol{R}_{\text{a}}^{\text{ext}}\right)'_X \\ \left(\boldsymbol{R}_{\text{b}}^{\text{int}}\right)'_X - \left(\boldsymbol{R}_{\text{b}}^{\text{ext}}\right)'_X \end{bmatrix} = \left\{ \begin{bmatrix} \boldsymbol{P}_{\text{aA}}^{\text{int}} & \boldsymbol{P}_{\text{aB}}^{\text{int}} \\ \boldsymbol{P}_{\text{bA}}^{\text{int}} & \boldsymbol{P}_{\text{bB}}^{\text{int}} \end{bmatrix} - \begin{bmatrix} \boldsymbol{P}_{\text{aA}}^{\text{ext}} & \boldsymbol{P}_{\text{aB}}^{\text{ext}} \\ \boldsymbol{P}_{\text{bA}}^{\text{ext}} & \boldsymbol{P}_{\text{bB}}^{\text{ext}} \end{bmatrix} \right\} \begin{bmatrix} \delta \boldsymbol{X}_{\text{A}} \\ \delta \boldsymbol{X}_{\text{B}} \end{bmatrix}, \tag{22}$$

the linearised form of the total variation of the residual $\boldsymbol{R}$ can be represented by

$$\boldsymbol{R}' = \left(\boldsymbol{K}^{\text{int}} - \boldsymbol{K}^{\text{ext}}\right) \delta \boldsymbol{v} + \left(\boldsymbol{P}^{\text{int}} - \boldsymbol{P}^{\text{ext}}\right) \delta \boldsymbol{X}. \tag{23}$$

The partitioned stiffness matrix $\boldsymbol{K}$ in Eq. (21) is equal to the matrix introduced in Eq. (5). The derivation of the stiffness matrix based on the internal and external part of the residual, as presented in Eq. (21), results in two contributions. On one hand, $\boldsymbol{K}^{\text{int}}$ contains the material and geometrical contribution to the stiffness, known from the variation of the internal part of the residual. On the other hand, $\boldsymbol{K}^{\text{ext}}$ is the so called *load correction matrix* and results from the variation of the external part of the residual. Initially, this term was proposed in [16] and [10] and has to be considered if external forces depend on the deformation themselves. Further explanations on theoretical background and numerical realisation can be found in [18, 20, 23] and [27, 26]. The overall stiffness matrix $\boldsymbol{K} = \boldsymbol{K}^{\text{int}} - \boldsymbol{K}^{\text{ext}}$ from Eq. (5) is subdivided in Eq. (21) in sub-matrices with the dimensions $\left\{\boldsymbol{K}, \boldsymbol{K}^{\text{int}}, \boldsymbol{K}^{\text{ext}}\right\} \in \mathbb{R}^{(n_\text{a}+n_\text{b})\times(n_\text{a}+n_\text{b})=n_v\times n_v}$ and therefore

$$
\begin{aligned}
\left\{\boldsymbol{K}_{\text{aa}}^{\text{int}}, \boldsymbol{K}_{\text{aa}}^{\text{ext}}\right\} &\in \mathbb{R}^{n_\text{a}\times n_\text{a}}, \qquad & \left\{\boldsymbol{K}_{\text{ab}}^{\text{int}}, \boldsymbol{K}_{\text{ab}}^{\text{ext}}\right\} &\in \mathbb{R}^{n_\text{a}\times n_\text{b}}, \\
\left\{\boldsymbol{K}_{\text{ba}}^{\text{int}}, \boldsymbol{K}_{\text{ba}}^{\text{ext}}\right\} &\in \mathbb{R}^{n_\text{b}\times n_\text{a}}, \qquad & \left\{\boldsymbol{K}_{\text{bb}}^{\text{int}}, \boldsymbol{K}_{\text{bb}}^{\text{ext}}\right\} &\in \mathbb{R}^{n_\text{b}\times n_\text{b}}.
\end{aligned}
$$

In Eq. (23) the quantities $\boldsymbol{P}^{\text{int}}$ and $\boldsymbol{P}^{\text{ext}}$ represent internal and external contributions to the overall pseudo load matrix $\boldsymbol{P} = \boldsymbol{P}^{\text{int}} - \boldsymbol{P}^{\text{ext}}$ from Eq. (6) and are subdivided in Eq. (22) to sub-matrices with the dimensions $\left\{\boldsymbol{P}, \boldsymbol{P}^{\text{int}}, \boldsymbol{P}^{\text{ext}}\right\} \in \mathbb{R}^{(n_\text{a}+n_\text{b})\times(n_\text{A}+n_\text{B})=n_v\times n_X}$ and therefore

$$
\begin{aligned}
\left\{\boldsymbol{P}_{\text{aA}}^{\text{int}}, \boldsymbol{P}_{\text{aA}}^{\text{ext}}\right\} &\in \mathbb{R}^{n_\text{a}\times n_\text{A}}, \qquad & \left\{\boldsymbol{P}_{\text{aB}}^{\text{int}}, \boldsymbol{P}_{\text{aB}}^{\text{ext}}\right\} &\in \mathbb{R}^{n_\text{a}\times n_\text{B}}, \\
\left\{\boldsymbol{P}_{\text{bA}}^{\text{int}}, \boldsymbol{P}_{\text{bA}}^{\text{ext}}\right\} &\in \mathbb{R}^{n_\text{b}\times n_\text{A}}, \qquad & \left\{\boldsymbol{P}_{\text{bB}}^{\text{int}}, \boldsymbol{P}_{\text{bB}}^{\text{ext}}\right\} &\in \mathbb{R}^{n_\text{b}\times n_\text{B}}.
\end{aligned}
$$

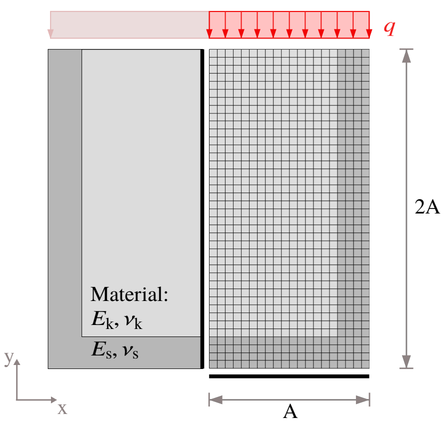
Using obtained sub-matrices, relation (19) for the sensitivity of reaction forces continues to

$$
\left(\boldsymbol{R}_{\text{b}}^{\text{ext}}\right)' = \boldsymbol{K}_{\text{ba}}^{\text{int}}\ \delta\boldsymbol{v}_\text{a} + \begin{bmatrix}\boldsymbol{P}_{\text{bA}}^{\text{int}} & \boldsymbol{P}_{\text{bB}}^{\text{int}}\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{X}_\text{A} \\ \delta\boldsymbol{X}_\text{B}\end{bmatrix} = \boldsymbol{K}_{\text{ba}}^{\text{int}}\ \delta\boldsymbol{v}_\text{a} + \boldsymbol{P}_{\text{b}}^{\text{int}}\,\delta\boldsymbol{X} = \left(\boldsymbol{K}_{\text{ba}}^{\text{int}}\ \boldsymbol{S}_\text{a} + \boldsymbol{P}_{\text{b}}^{\text{int}}\right)\delta\boldsymbol{X}, \quad (24)
$$

where the contributions to the pseudo load matrix are summarised in $\boldsymbol{P}_{\text{b}}^{\text{int}} = \begin{bmatrix}\boldsymbol{P}_{\text{bA}}^{\text{int}} & \boldsymbol{P}_{\text{bB}}^{\text{int}}\end{bmatrix}$. Here, the relation (16) for the sensitivity of the state variable $\delta\boldsymbol{v}_\text{a}$ is used. The sensitivity relation in Eq. (24), particularly the quantity which corresponds to the partial derivative of the external part of the residual on boundary Γ_D with respect to design $\partial\boldsymbol{R}_{\text{b}}^{\text{ext}}/\partial\boldsymbol{X} = \boldsymbol{K}_{\text{ba}}^{\text{int}}\ \boldsymbol{S}_\text{a} + \boldsymbol{P}_{\text{b}}^{\text{int}}$, can be implemented into an existing framework for structural optimisation. The reaction forces $\boldsymbol{R}_{\text{b}}^{\text{ext}}$ as well as their sensitivities $\left(\boldsymbol{R}_{\text{b}}^{\text{ext}}\right)'$ can be used as objective or constraint functional within the posed optimisation Problem 1. Details on the explicit formulations for $\boldsymbol{R}$, $\boldsymbol{K}$ and $\boldsymbol{P}$ can be found in [12], for instance. Analytically derived sensitivities can be verified using the *finite difference method* witch can be applied to prove the accuracy of provided sensitivity information.

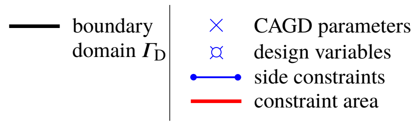
4 Applications and numerical investigations

In the following example close attention is paid to the sensitivity analysis of physical reaction forces, which are used as indicators for the design of support areas. Computations for compliance and volume minimisation with the reduction of the resulting maximum amplitude of reaction forces to a certain prescribed maximum value using inequality constraints accentuate the possible usage within stated optimisation problems. According to the sketch in Fig. 1, a multi-material domain with the side length A, loaded by a surface load on the top, fixed on the ground, and



Model parameters

A	1.0
n_{elx}	20
n_{ely}	40
E_s	25000
E_k	$0.5 \cdot E_s$
$\nu_s = \nu_k$	0.20
t	0.10
q	25.0



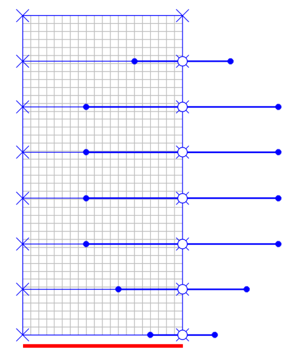


Figure 1: Domain with multi-material: mechanical system, FE-mesh, boundary conditions and load case, model parameters and optimisation model (design variables, side constraints and constraints area)

two different materials, an outer shell material E_s and a kernel material E_k, is investigated. Due to symmetry, half of the mechanical system is considered. The finite element analysis model is also illustrated in Fig. 1. The overall number of elements $n_{el} = 800$ for the FE-mesh results from the choice of the number of elements in x- and y-direction ($n_{elx} = 20, n_{ely} = 40$). For a pure displacement and geometrically non-linear element formulation based on two displacement degrees of freedom per node and Neo-Hookean constitutive law, the overall number of degrees of freedom results to $n_v = 1722$. All investigations are based on the model parameters listed in Fig. 1. It also contains the underling optimisation model, where a CAGD-model with 16 control points assembling one Bézier patch is used for the geometry description. The number of design parameters on the nodal basis amounts $n_X = 1722$ and on the geometry basis $n_{cp} = 32$. The final subset of design variables used for optimisation counts $n_s = 7$ design parameters and is reselected from the set of parameters for the described geometry model. Hints on the choice of design variables in terms of nodal and geometry design parameters are given in Remark 2.2. Referring the formulation of an optimisation Problem 1, objectives and constraints are specified in following subsections. Here, only lower and upper side constraints are introduced

$$\begin{bmatrix}0.7 & 0.4 & 0.4 & 0.4 & 0.4 & 0.6 & 0.8\end{bmatrix}^T = \boldsymbol{s}^{\mathrm{l}} \leq \boldsymbol{s} \leq \boldsymbol{s}^{\mathrm{u}} = \begin{bmatrix}1.3 & 1.6 & 1.6 & 1.6 & 1.6 & 1.4 & 1.2\end{bmatrix}^T. \tag{25}$$

They are valid for subsequent investigations and are arranged in descending order from top to bottom according to the optimisation model in Fig. 1. The surface load $q = 25.0$ is distributed on the present 11 nodes on the top equivalently and results in nodal forces of $F_{\mathrm{n}} = 2.5$ for regular and of $F_{\mathrm{nc}} = 1.25$ for corner nodes.

4.1 Compliance minimisation and reduction of forces

In this study, the compliance minimisation problem with the objective $J = C = \boldsymbol{F}^T \boldsymbol{v}$ is investigated. The physical reaction forces in the constraint area (cf. Fig. 1) have to be reduced to the maximum amplitude of 75% compared to the amplitude of reaction force for the initial design. This can be done by the incorporation of the reaction forces as inequality constraints

$g = F_{\mathrm{R}}$ and the definition of $F_{\mathrm{R,max}}$ as the maximum value. Used side constraints $s^{\mathrm{l}}, s^{\mathrm{u}}$ are presented in Eq. (25). The mathematical optimisation algorithm used 11 iterations to obtain a minimum value for the objective, which could be reduced by approximately 6% compared to the initial design. In parallel, the incorporation of reaction forces as constraints gives the advantage and the possibility to reduce them too. Both results are presented in Fig. 2. The optimal distribution of design variables, which are all in the prescribed boundaries or side constraints $s^{\mathrm{l}}, s^{\mathrm{u}}$ is pictured in Fig. 3. Here, the contour of the initial profile of reaction forces is compared to the profile of reaction forces for the optimised design. Finally, it is the engineer's or designer's

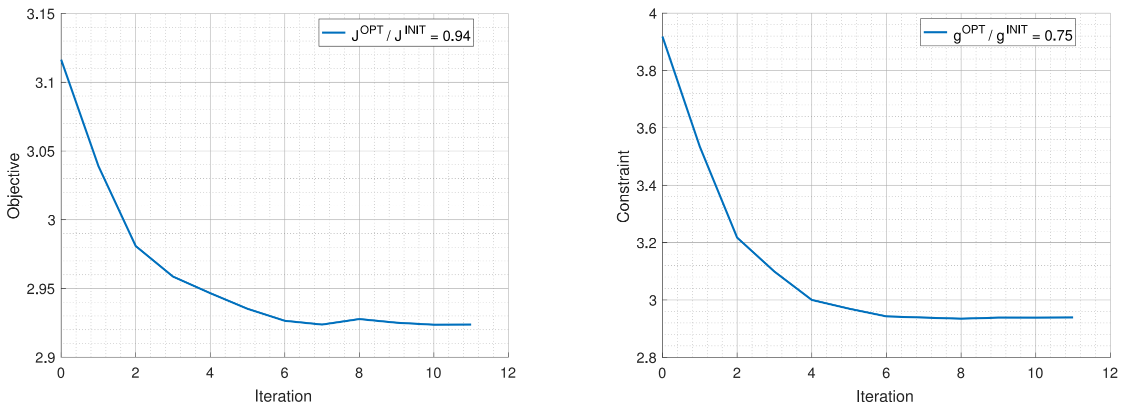


Figure 2: Optimisation results (objective and constraint) over iterations

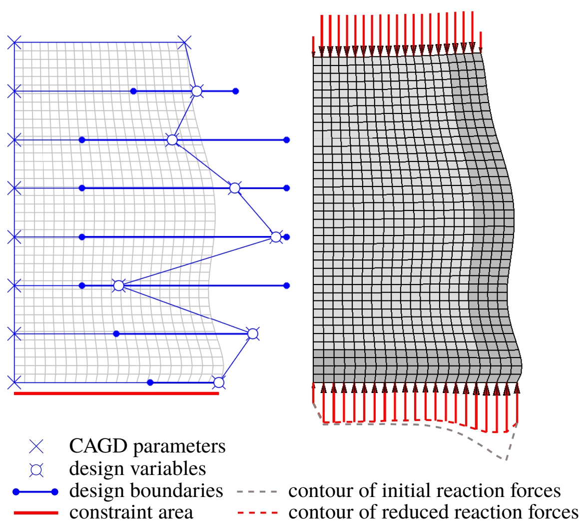


Figure 3: Optimal design and comparison of reaction forces for the deformed system (scaling($\boldsymbol{v}$) = 1.0)

choice how to manage the balance between compliance minimisation and reduction of reaction forces. If the required maximum value for reduction of 75% is increased, the balance in the overall potential of the system between objective and constraint will change and the compliance minimisation will lead to higher reduction values than only 6%.

4.2 Volume minimisation and reduction of forces

Here, the setup for the optimisation problem is similar to the setup for the compliance minimisation presented in Section 4.1. The only difference is that the objective function will be exchanged and the overall volume $J = V$ of the given system has to be minimised. The maximum amplitude of reaction forces has to be reduced to 75% compared to the initial design by incorporation of inequality constraints and a maximum value $F_{\mathrm{R,max}}$. The optimisation algorithm reaches the optimum value for the objective after 11 iterations and the optimisation process is aborted. The overall volume can be reduced by approximately 37% compared to the initial design. Furthermore the constraint for the reaction forces is fulfilled and allows to reduce them by 25% compared to the initial design. These results are illustrated in Fig. 4. The corresponding distribution of design variables which remain in the prescribed side constraints $\boldsymbol{s}^{\mathrm{l}}$, $\boldsymbol{s}^{\mathrm{u}}$ from Eq. (25) as well as the contour of the initial and optimised profile of reaction forces is presented in Fig. 5. In this case, the designer or the engineer also has to decide on the balance between volume minimisation and reduction of reaction forces. For the chosen and presented optimisation setup, the large resulting displacements on the right top side of the system (cf. Fig. 5) are the consequence. If the system or the material is able to handle this kind of displacement amplitude, the gain is the enourmous reduction of volume and the reduction of final reaction forces as pressure loads on the ground.

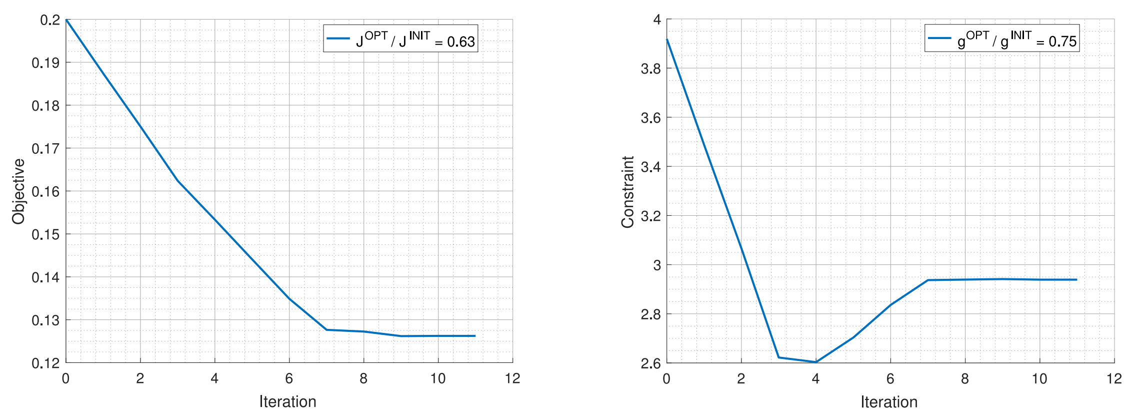


Figure 4: Optimisation results (objective and constraint) over iterations

5 Conclusion

Based on the variational approach for design sensitivity analysis, and especially on the intrinsic formulation with enhanced kinematics, the sensitivity relation for physical reaction forces was derived from available tangent operators and the sensitivity relation for the state. The introduced fundamental relations were partitioned into contributions in the inner domain and on two types of boundaries. This decomposition allowed to divide resulting tangent operators in necessary contributions for the required sensitivity information. The obtained gradient information was investigated concerning compliance and volume minimisation. Two studies accentuated possible usage and application of the sensitivity information, for instance how to control amplitudes of arising forces which interact with foundations of mechanical parts.

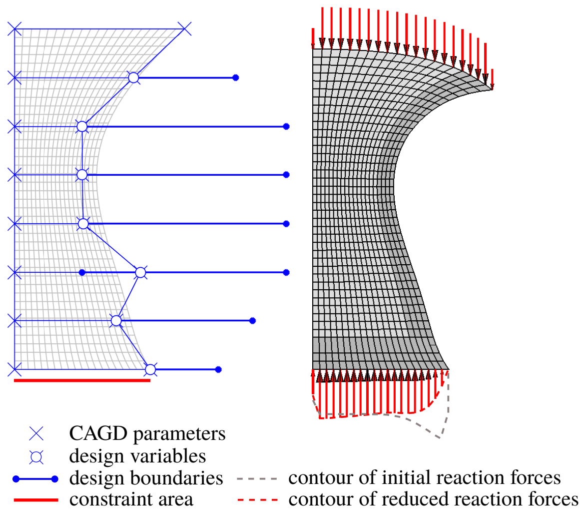


Figure 5: Optimal design and comparison of reaction forces for the deformed system (scaling($\boldsymbol{v}$) = 1.0)

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