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AN EXTENSION OF ALGEBRAIC EQUATIONS OF ELASTIC TRUSSES WITH SELF-EQUILIBRATED SYSTEM OF FORCES

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Abstract. Linear elastic analysis of truss structures can be done within the finite element method formalism as well as without the approximation of the displacement field, by algebraic equations. The present paper is an extension of the considerations presented in [1] to the algebraic equations for geometric stiffness matrix. The matrix allow to include the influence of self-equilibrated systems of forces on the response of truss structure. It is a crucial aspect for the qualitative and quantitative analyses of tensegrity-like trusses.

1 INTRODUCTION

Trusses are an important class of Structural Mechanics tasks. Static analysis of trusses in a linearly elastic range can be carried out by means of equations obtained using the finite element method (FEM) [2], which dominates the computational market. The truss equations can also be derived by omitting the typical for FEM approximation of displacements by means of algebraic equations (see [3]). The synthesis of the algebraic formulation, along with the extension to the analysis of frames and grillages, was carried out in [1].

In trusses there are frequently self-equilibrated systems of axial forces that meet identically homogeneous systems of equilibrium equations. Such a situation occurs in the qualitative and quantitative analysis of tensegrity lattice structures [4, 5]. This paper supplements the considerations given in [1] with the algebraic form of expressions on the geometric stiffness matrix, which allows analyzing the influence of self-balancing longitudinal forces on the work of the structure. This formulation is of significant importance from the scientific (optimization, tensegrity, convex sets, etc.) as well as didactic point of view.

2 GENERAL FORMULATION

Let us to consider a plane pin-joint structure composed of e straight and prismatic bars of the lengths l_k, cross sections A_k and Young modulus E_k. The bars are connected in nodes in which the number of s nodal displacements q_j and nodal forces Q_i are defined [1]. Axial forces N_k can be expressed by the extensions of bars Δ_k in the form

$$N_k = \frac{E_k A_k}{l_k}\Delta_k, \quad k = 1, 2, ..., e. \tag{1}$$

The extensions Δ_k are a combination of nodal displacements

$$\Delta_k = \sum_{j=1}^{s} B_{kj} q_j, \quad j = 1, 2, ..., s. \tag{2}$$

The extension matrix $[B_{kj}]$ can be defined by the projection of nodal displacements on the bar axes. Additionally the self-equilibrated system of axial forces S_k which satisfy the homogeneous set of equilibrium equations

$$\sum_{k=1}^{e} B_{jk} S_k = 0 \tag{3}$$

is considered. The self-equilibrated forces as well as possible mechanisms of the structure can be found with the use of singular value decomposition (SVD) of the matrix $[B_{ik}]$ [6]. If one consider equations of equilibrium in the actual configuration then moment

$$M_k = S_k l_k \psi_k \tag{4}$$

is acting on each bar. Angles of bar rotations ψ_k can be expressed as a combination of nodal displacements

$$\psi_k = \frac{1}{l_k}\sum_{j=1}^{s} C_{kj} q_j. \tag{5}$$

The matrix $[C_{kj}]$ can be defined by the projection of nodal displacements on the direction perpendicular to bar axes with right-hand positive signs of rotations.

The above formalism leads to the linear system of algebraic equations

$$\sum_{j=1}^{s} (k_{ij} + k_{ij}^{G}) q_j = Q_i \tag{6}$$

in which the stiffness matrix k_{ij} and geometric stiffness matrix k_{ij}^G can be experssed in algebraic form

$$k_{ij} = \sum_{k=1}^{e} B_{ki}\frac{E_k A_k}{l_k} B_{kj}, \tag{7}$$

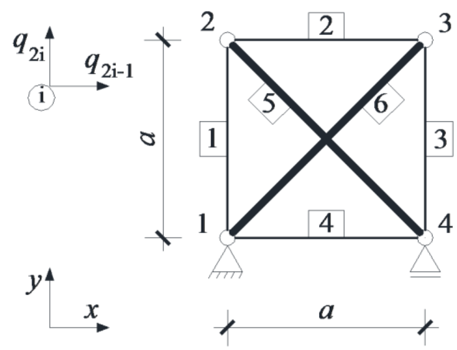

Figure 1: The "X" truss

$$k_{ij}^G = \sum_{k=1}^{e} C_{ki} \frac{S_k}{l_k} C_{kj}. \tag{8}$$

The above considerations can be relative easily extended for 3D truss structures.

Let us define the p-th row of the matrix B_{kj} by $\mathbf{b}_p$ and p-th row of the matrix C_{kj} by $\mathbf{c}_p$. The stiffness and geometric stiffness matrices can be decomposed in the form

$$\mathbf{K} = \sum_{p=1}^{e} \mathbf{K}^{(p)}, \quad \mathbf{K}^{(p)} = \frac{E_p A_p}{l_p} \mathbf{b}_p \otimes \mathbf{b}_p, \tag{9}$$

$$\mathbf{K}^G = \sum_{p=1}^{e} \mathbf{K}^{G(p)}, \quad \mathbf{K}^{G(p)} = \frac{S_p}{l_p} \mathbf{c}_p \otimes \mathbf{c}_p, \tag{10}$$

where $\mathbf{a} \otimes \mathbf{b} = [a_i b_j]$ is a dyadic product of two vectors $\mathbf{a}$ and $\mathbf{b}$. The expression for $\mathbf{K}$ was introduced in [1] and the expression for $\mathbf{K}^G$ is original. The order of matrices $\mathbf{K}^{(p)}$ and $\mathbf{K}^{G(p)}$ is one. This kind of decomposition can be successfully used in optimization [7] or in the uncertainity analysis with the use of convex sets [8].

3 EXAMPLE

The subject under consideration is six element "X" truss with five nodal displacements (Fig. 1).

Let us introduce the following matrices

$$\mathbf{B} = [B_{ik}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix},$$

$$\mathbf{C} = [C_{ik}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix},$$

$$\mathbf{E} = \operatorname{diag}\left[\frac{E_i A_i}{L_i}\right] = \operatorname{diag}\left[\frac{E_1 A_1}{a}, \frac{E_2 A_2}{a}, \frac{E_3 A_3}{a}, \frac{E_4 A_4}{a}, \frac{E_5 A_5}{a\sqrt{2}}, \frac{E_6 A_6}{a\sqrt{2}}\right] =$$
$$= \frac{EA}{a}\operatorname{diag}\left[1, 1, 1, 1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right].$$

The self-equilibrated set of forces is obvious but can be also found by the SVD decomposition of the extension matrix B_{kj}. There are no mechanism in the structure.

$$\mathbf{S} = \operatorname{diag}\left[\frac{S_i}{L_i}\right] = \operatorname{diag}\left[\frac{S_1}{a}, \frac{S_2}{a}, \frac{S_3}{a}, \frac{S_4}{a}, \frac{S_5}{a\sqrt{2}}, \frac{S_6}{a\sqrt{2}}\right] =$$
$$= \operatorname{diag}\left[\frac{S}{a}, \frac{S}{a}, \frac{S}{a}, \frac{S}{a}, \frac{-S}{a}, \frac{-S}{a}\right] = \frac{S}{a}\operatorname{diag}\left[1, 1, 1, 1, -1, -1\right].$$

Global stiffness and geometric stiffness matrices, according to the formulae (7) and (8), are the following

$$\mathbf{K} = [k_{ik}] = \frac{EA}{4a}\begin{bmatrix} 4+\sqrt{2} & -\sqrt{2} & -4 & 0 & -\sqrt{2} \\ -\sqrt{2} & 4+\sqrt{2} & 0 & 0 & \sqrt{2} \\ -4 & 0 & 4+\sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} & 4+\sqrt{2} & 0 \\ -\sqrt{2} & \sqrt{2} & 0 & 0 & 4+\sqrt{2} \end{bmatrix},$$

$$\mathbf{K}^G = \left[k^G_{ik}\right] = \frac{S}{2a}\begin{bmatrix} 1 & -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & -2 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 & 1 \end{bmatrix}.$$

Equivalent matrices can be derived through the Finite Element Method [2, 5]. If the structure on Fig. 1 is divided into six finite elements with the node numbers 1-2, 2-3, 3-4, 1-4, 2-4, 1-3 and the element lengths $L_1 = L_2 = L_3 = L_4 = a$, $L_5 = L_6 = a\sqrt{2}$ the following matrices are to be defined on each finite element level:

$$\mathbf{k}_e + \mathbf{k}^G_e = \frac{E_e A_e}{L_e}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + \frac{S_e}{L_e}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}, \quad e = 1, ..., 6,$$

$$\mathbf{d}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{d}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{d}_3 = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{d}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{d}_5 = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 & 0 & -\frac{\sqrt{2}}{2} \end{bmatrix}, \quad \mathbf{d}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}.$$

Homogeneous boundary conditions are already included in the FEM matrix formulation. Global matrices can be derived as $\mathbf{K}+\mathbf{K}^G = \sum_{e=1}^{6} \mathbf{d}_e^T \left(\mathbf{k}_e + \mathbf{k}_e^G\right) \mathbf{d}_e$ to obtain exactly the same matrices like in the algebraic formulation presented above. The matrices can be also decomposed with the formulae (9) and (10) in the following form

$$\mathbf{K} = \frac{E_1A_1}{a} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{E_2A_2}{a} \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{E_3A_3}{a} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} +$$

$$+ \frac{E_4A_4}{a} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + \frac{E_5A_5}{2a} \begin{bmatrix} 1 & -1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix} + \frac{E_6A_6}{2a} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{K}^G = \frac{S_1}{a} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{S_2}{a} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{S_3}{a} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} +$$

$$+ \frac{S_4}{a} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \frac{S_5}{2a} \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix} + \frac{S_6}{2a} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4 CONCLUSIONS

The present paper is an extension of [1] for the algebraic form of the geometric stiffness matrix for the self-equilibrated systems of normal forces in truss structures. Approximation of the unknown displacement field is not required. The results are the same as in the finite element method. Decomposition of the geometric stiffness matrix is proposed as a sum of dyadic product of the rows of extension matrix multiplied by the stiffness of bars.

The form is suitable for the optimization as well as in the uncertainity analysis with the use of convex sets.

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