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FORCE METHOD DEVELOPMENT IN STRUCTURAL MECHANICS FOR THE NONLINEAR TASKS. FEM HYBRID MODELS

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Abstract. For the forces method development in the nonlinear field the generalized More’s formula with tangent stiffness matrix has been invented. This matrix was obtained as the integral parameter of the stress-strained condition of the whole cross-section rod points. Formulation of the finite elements method is considered as a movement method and a forces method. The advantages and disadvantages of the two forms of the FEM in elastic-plastic analysis noted. It is proposed to form the procedure of calculating bending rod systems via the force method based on the bending moments formation by using state equations.
There is shown the key differences and advantages at calculation of rod systems by the discrete-analytical method. The results of calculation of rod systems obtained by the discrete-analytical method with the finite element method are compared. The proposed method allows to obtain adequate results without significant processor and time consuming.

1 INTRODUCTION

In modern engineering practice it’s often required to solve tasks connecting with a determination of elastoplastic deformations. Nowadays it is possible to complete a nonlinear analysis only by using powerful computer programs based on application of the finite element method (FEM). While solving nonlinear tasks, connecting with analysis of elastoplastic process in constructions, by using the finite element method the processing time increases acutely, that is often unacceptable in project conditions.

In most cases for calculation of building structure the rod systems, covering a wide range of engineering tasks, are used. For such kind of systems it is possible to calculate the elastoplastic tasks via generalization of classical methods of construction mechanics – force method and displacement method. This generalization involves the using of an explicit time-based computational scheme and the determination at each step of the system tangential stiffness. The approach based on the generalized method of forces (generalized displacement method) allows drastically reduce the time for solving elastoplastic tasks.

The aim of this research was to develop a discrete-analytical method for calculating rod systems in the nonlinear area. Calculation algorithms for plane and spatial rod systems are proposed. The calculation results are compared with FEM.

The scientific rationalization and difference from the finite element method (FEM) are in the exclusion of finite elements during making an algebraic equations' system and their substitution by the sections on which the dispensing stiffness parameters are integrated. As a result, the number of system equations will correspond to the number of static (or kinematic) indeterminations of the rod system. Moreover if the rod system is statically determined there is no need to solve the equations' system. In this case any stiffness parameters allocation along the rod's length is considered using the tangential stiffness matrix and generalized More's formula. Obtained hybrid scheme, in a way, can be compared with the boundary element method. However the integration should be completed here not across the entire area, but within the long element-rod. The nonlinear calculation algorithm by the discrete-analytical method for determination of internal forces in nodes of statically indeterminate rod system will be considered further. Herein a preference is given to the displacement method due to the simplicity of the main system choice, the way of equilibrium equations construction, compilation of the stiffness matrix and the external loads vector. The force method with the generalized More's formula jointly are used to determine the rod stiffness coefficients, in consideration with the nonlinear action.

The force method has not yet been realized in full measure due to the difficulties in the making of canonical equations of deformation compatibility for the system as a whole. Herewith there is an undeniable advantage in the classical force method due to the lack of necessity to determine the rod local stiffness matrix. Initially it was supposed to use the force method to improve the efficiency of nonlinear analysis. According to research it turned out that FEM form in form of displacement method form is more preferably for a particular stepper method. It is proposed to consider the pros and cons of two FEM forms in elastoplastic analysis.

Advantages of the forces method:
- an accuracy of stresses' determination;
- a direct creation of flexibility matrix;
- account of absolutely rigid rods.

Disadvantages of the forces method:
- more unknowns at rigid several rods' connection;
- additional expressions are needed to determine full movements;
- difficulties in making an equations' system and accounting for the geometric nonlinearity.

Advantages of the displacement method:
- less amount of unknowns at rigid rods' connection;
- natural mechanism of making a resolving equations' system;
- direct getting of displacement in solving the equations' system;
- accounting for the geometric nonlinearity.

Disadvantages of the movement method:
- inaccuracy in stresses' determining as they are secondary;
- complexity in accounting for absolutely rigid rods;
- necessity to determine the interim stiffness matrix. But due to the rod stiffness matrix is available to determine through the shape functions it is not necessary to calculate an interim equations system to determine the stiffness coefficients in the nodes.

Theoretically a new resolving equations' system in the form of the forces method should be obtained based on Castigliano's variational concept, but there are some difficulties appeared in practice and it is necessary to develop a special mechanism for making this system. Thus, the most appropriate way to optimize nonlinear analysis is a hybrid approach in which the FEM in displacement method form is used to determine the internal forces in the long rod nodes, to account for the distribution of internal forces along the rod's length – integral analytical expressions. In this case the final elements are replaced by integration parts thereby leading to a significant calculation acceleration. It is proposed to consider this method thoroughly.

2 GENERAL RELATIONS

2.1 PLANE FRAMEWORK

It is proposed to consider a hybrid method for determining elastoplastic deformations in rod systems. The nonlinear analysis scheme consists of the following steps:

- determination of the rod's stiffness matrix considering the distributed parameters in length, using the generalized More's formula;
- bringing a stress distribution to the nodes and recalculation in level with geometric nonlinearity;
- making an equations' system for determining the movements in the rods' nodes;
- reactions' determination of the rods' nodes;
- definition of the function of the internal stresses changes in curved rods';
- stresses' determination in the integration points using the generalized curvature;
- definition of the function of the movements' changes along the rod's length;
- definition of the tangent stiffness matrix, describing the stiffness in the rods' cross section.

For the plane task, the tangent stiffness matrix can be defined using integral function of the cross-section's state law [1, 2]. There is shown a formula for determining the shear stiffness of a rectangular section [3]:

$$T(\tau)=\begin{cases}\dfrac{\sigma_S bh^3}{4\varepsilon_S}\cdot\dfrac{1}{3},\tau\le 1\\ \dfrac{\sigma_S bh^3}{4\varepsilon_S}\cdot\dfrac{1}{3\tau(x)^3}\cdot\left[1+a\left(\tau(x)^3-1\right)\right],\tau>1\end{cases},a=\frac{E_{pl}}{E}, \tag{1}$$

$$\tau=\frac{\chi h}{2\varepsilon_s}, \tag{2}$$

where ε_s – deformation corresponding to the yield strength, σ_s – yield strength, χ – the rod's curvature in the considered cross section, b, h – parameters of rectangular cross section; E_{pl} – tangent (plastic) modulus of elasticity.

Inasmuch the shift movement is ignored here, the rod's stiffness matrix can be obtained via form functions (Fig. 1)

$$N_1=1-\frac{x}{l};N_2=(1-\frac{3x^2}{l^2}+\frac{2x^3}{l^3});\ N_3=(x-\frac{2x^2}{l}+\frac{x^3}{l^2}), \tag{3}$$

$$N_4 = \frac{x}{l};\; N_5 = (\frac{3x^2}{l^2} - \frac{2x^3}{l^3});\; N_6 = (-\frac{x^2}{l} + \frac{x^3}{l^2}). \tag{4}$$

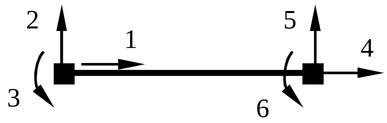

Figure 1: Degrees of freedom

Differentiating the potential deformation energy by movements, a formula for determining the stiffness matrix of the bending rod element is obtained:

$$\left[K^e\right] = EI\int_0^l \frac{d^2[N]^{\mathrm{T}}}{dx^2}\frac{d^2[N]}{dx^2}dx = EI\int_0^l \begin{bmatrix} N_2'' \\ N_3'' \\ N_5'' \\ N_6'' \end{bmatrix}\begin{bmatrix} N_2'' & N_3'' & N_5'' & N_6'' \end{bmatrix} dx = EI\int_0^l \begin{bmatrix} N_2''N_2'' & N_2''N_3'' & N_2''N_5'' & N_2''N_6'' \\ N_3''N_2'' & N_3''N_3'' & N_3''N_5'' & N_3''N_6'' \\ N_5''N_2'' & N_5''N_3'' & N_5''N_5'' & N_5''N_6'' \\ N_6''N_2'' & N_6''N_3'' & N_6''N_5'' & N_6''N_6'' \end{bmatrix} dx \tag{5}$$

Replacing EI with $T(\tau)$ and entering under the integral sign. Considering the axial stiffness coefficients, a formula for determining the stiffness matrix on step by the form functions is obtained.

$$\left[K^e\right] = \int_0^l \begin{bmatrix} \frac{F}{I}\cdot N_1'N_1' & 0 & 0 & \frac{F}{I}\cdot N_1'N_4' & 0 & 0 \\ 0 & N_2''N_2'' & N_2''N_3'' & 0 & N_2''N_5'' & N_2''N_6'' \\ 0 & N_3''N_2'' & N_3''N_3'' & 0 & N_3''N_5'' & N_3''N_6'' \\ \frac{F}{I}\cdot N_4'N_1' & 0 & 0 & \frac{F}{I}\cdot N_4'N_4' & 0 & 0 \\ 0 & N_5''N_2'' & N_5''N_3'' & 0 & N_5''N_5'' & N_5''N_6'' \\ 0 & N_5''N_2'' & N_6''N_3'' & 0 & N_6''N_5'' & N_6''N_6'' \end{bmatrix} \cdot T(\tau)\,dx, \tag{6}$$

where F – sectional area; I – the moment of cross section's inertia.

Bringing the stress distribution p to equivalent on step

$$\left\{P^e\right\} = \int_0^l [N]^{\mathrm{T}}\, p \cdot \frac{T(\tau)}{EI}\,dx = p\int_0^l \begin{bmatrix} N_2 \\ N_3 \\ N_5 \\ N_6 \end{bmatrix} \cdot \frac{T(\tau)}{EI}\,dx\ . \tag{7}$$

After the stiffness ratios k_{ij} have been determined the stiffness matrix in the global coordinate system is defined in the ordinary way at each step. The stiffness ratios converging in the rods' node are combained. Then in incremental form for stress increasing at step the equations of the finite element method are solved for the system as a whole and the moments' increments in sections are defined $\Delta M(x)$.

The increments definition of curvature along the rods length:

$$\Delta\chi(x) = \frac{\Delta M(x)}{T(\tau)}, \tag{8}$$

The deformation definition in extreme fibers:

$$\Delta\varepsilon(x) = \Delta\chi(x)\frac{h}{2}. \tag{9}$$

The stress and deformation definition at step:

$$\varepsilon(x) = \varepsilon(x) + \Delta\varepsilon(x) \tag{10}$$

$$\sigma(x) = \begin{cases} \sigma(x) + E_{pl}\Delta\varepsilon(x), \tau > 1 \\ \sigma(x) + E\Delta\varepsilon(x), \tau \le 1 \end{cases}. \tag{11}$$

Further a new local stiffness matrix of the rod element is developed via the integration (6), considering the change in stiffness in the cross-section. And the cycle repeats.

The geometric nonlinearity record can be done using formulas [2]:

$$u(x) = \int_x^l \chi(x) \cdot x dx\,, \tag{12}$$

$$\theta(x) = \int_x^l \chi(x) dx\,, \tag{13}$$

$$x = cos(\theta) \cdot x_0\,. \tag{14}$$

2.2 SPATIAL FRAMEWORK

The forces method was used to calculate the spatial framework during the stiffness matrix obtaining, considering the shear displacements. The tangent stiffness matrix in conditions of elastoplastic deformation with shear displacements is determined by the formula [4, 5]:

$$[T] = \int_F [L][A][S] dF\,, \tag{15}$$

$$[L] = \begin{bmatrix} \eta & 0 & 0 \\ -\xi & 0 & 0 \\ 0 & \xi & -\eta \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, [A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, [S] = \begin{bmatrix} 0 & -\xi & \eta & 1 & 0 & 0 \\ \xi & 0 & 0 & 0 & 0 & 1 \\ -\eta & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \tag{16}$$

where A – matrix with pseudoelastic ratios describing the connection eqiuations between stress and deformation [5]; S – matrix with section coordinates (Fig. 2) for the connection eqiuations between the deformation velocity and kinematic parameters [4]; L – transition matrix from stresses to internal forces:

$$\int_F [L]\{\dot{\sigma}\} dF = \left\{\frac{\partial' M}{\partial t}\right\}, \tag{17}$$

$$\left\{\frac{\partial' M}{\partial t}\right\} = [T]\{\dot{\Psi}\}, \tag{18}$$

$$\left\{\frac{\partial' M}{\partial t}\right\}=\left[\frac{\partial' M_1}{\partial t} \quad \frac{\partial' M_2}{\partial t} \quad \frac{\partial' M_3}{\partial t} \quad \frac{\partial' Q_1}{\partial t} \quad \frac{\partial' Q_2}{\partial t} \quad \frac{\partial' Q_3}{\partial t}\right]^T, \tag{19}$$

where $M_1, M_2, M_3, Q_1, Q_2, Q_3$ – vectors components M and Q in the coordinate system ζ, η, ξ.

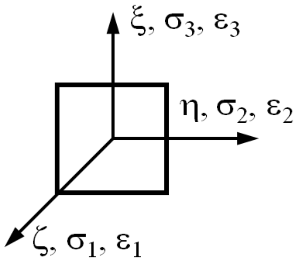

Figure 2: Coordinates of section

The generalized More’s formula is written for determination of displacements at time step.

$$\left[\dot{\Delta}\right]=\int_0^l [u_e]\cdot\left\{\dot{\Psi}\right\}dx=\int_0^l [u_e]\cdot[T]^{-1}\left\{\frac{\partial' M}{\partial t}\right\}dx, \tag{20}$$

$$\left[\dot{\Phi}\right]=\int_0^l [u_{em}]\cdot[T]^{-1}\left\{\frac{\partial' M}{\partial t}\right\}dx, \tag{21}$$

$$[u_e]=[M_{1e} \quad M_{2e} \quad M_{3e} \quad Q_{1e} \quad Q_{2e} \quad Q_{3e}], \tag{22}$$

$$\left[\dot{\Psi}\right]=\left[\dot{\chi}_1 \quad \dot{\chi}_2 \quad \dot{\chi}_3 \quad \frac{\partial w}{\partial x} \quad \dot{\gamma}_3 \quad -\dot{\gamma}_2\right]^T, \tag{23}$$

where $M_{1e},\ldots, Q_{1e},\ldots$ – the vectors components of the moments and internal forces M_e, Q_e in the coordinate system ξ, η, ζ from the application of a single force and a single moment; u_{em} – the matrix-line with a single moment; $\partial'\chi/\partial t$– the changes vector in time of the generalized curvatures; γ'_2, γ'_3 – the shear velocity cross-section; $\partial w/\partial x$ – velocity in axial direction.

Using a generalized More’s formula, the flexibility matrix for cantilever beam is determined by formula:

$$[\delta]=\int_0^l [u_e]\cdot[T]^{-1}[u_e]^T dx, \tag{24}$$

6 equations systems with dimension 6x6 are solved gradually via forces method, changing single movement directions Δ_e

$$\{K_\Delta\}=[\delta]^{-1}\{\Delta_e\}. \tag{25}$$

Considering the symmetry, the rod’s stiffness matrix K^e is obtained at step dimension 12x12. Then the system stiffness matrix is made in ordinary way. As a result of solving the resolving equations system, the movements in the framework nodes will be obtained:

$$\{\Delta\} = [K]^{-1}\{P\}. \tag{26}$$

The movement of each rod nodes in global axes:

$$\{\Delta\} \to \{\Delta_g^e\}. \tag{27}$$

The movement of rod nodes in the local coordinate system using a transformation matrix *R* is determined via formula:

$$\{\Delta^e\} = [R]\{\Delta_g^e\}. \tag{28}$$

Reactions in the rod nodes are determined via formula:

$$\{P^e\} = [K^e]\{\Delta^e\}, \tag{29}$$

$$\{P^{en}\} = \begin{Bmatrix} \{M^n\} = [M_1^n \quad M_2^n \quad M_3^n]^T \\ \{Q^n\} = [Q_1^n \quad Q_2^n \quad Q_3^n]^T \end{Bmatrix}. \tag{30}$$

It is written the expressions for functions of internal forces and moments change, using (30):

$$\{M^e(x)\} = [Q^n] \times \{r(s)\} - \{M^n\}, \tag{31}$$

$$\{Q^e(x)\} = \{Q^n\}, \tag{32}$$

and it is presented as a vector:

$$\{u^e(x)\} = [M_1^e(x) \quad M_2^e(x) \quad M_3^e(x) \quad Q_1^e(x) \quad Q_2^e(x) \quad Q_3^e(x)]^T. \tag{33}$$

Then it is defined the rod section curvature via formula:

$$\{\Psi^e\} = [T^e]^{-1}\{u^e\}. \tag{34}$$

As a result, it is obtained a stress vector describing the stress state at each point of the rod section at the time step:

$$\{\sigma^e\} = [A]^e[S]\{\Psi^e\}. \tag{35}$$

$$\{\sigma^e\} = [\sigma_1 \quad \sigma_2 \quad \sigma_3]^T. \tag{36}$$

$$\sigma_1 = \sigma_{\zeta\zeta}; \sigma_2 = \tau_{\eta\zeta}; \sigma_3 = \tau_{\xi\zeta}. \tag{37}$$

For the stress state in the rod $\sigma_{\eta\eta} = \sigma_{\xi\xi} = \tau_{\xi\eta} = 0$.

Transitions from elastoplastic stress to elastic one and back are determined by logical conditions:

- elastic stress $\sigma^2_{sum} \le \sigma^2_t$
- elastoplastic stress $\sigma^2_{sum} > \sigma^2_t$, $d\sigma^2_{sum}/dt > 0$

$$\sigma_{sum}^2 = \sigma_1^2 + 3\sigma_2^2 + 3\sigma_3^2. \tag{38}$$

Via σ_t the value of the total stress at the beginning of the n-th semicircle of elastic loading is indicated (if $t = 1$, that σ_t will be equal the yield strength σ_s).

Matrix components *S* depend on coordinates of considering rod section point. Matrix components A depend on stress at section point at each time step. During elastoplastic deformation the matrix A corresponds to the differential analogue of Hooke's law with pseudoelastic ratios depending on the stress state at the point [5]. When creating computational algorithms at each time step, the abovementioned formulas are written as incremental ratios.

Write down expressions for determining the stiffness ratios of the rod taking into account the geometric nonlinearity.

The function of changes in internal stress:

$$\{Q^e(x)\} = \{Q^n\} \times \{e(x)\}. \tag{39}$$

Unit vector:

$$\{e(x)\} = \left\{\frac{dr(x)}{dx}\right\}. \tag{40}$$

Radius-vector:

$$\{r(x)\} = \left\{\frac{dr(x)}{dt}\right\} \cdot \Delta t \ , \tag{41}$$

$$\left\{\frac{dr(x)}{dt}\right\} = \int_0^l [u_{e1e2e3}]\{\dot{\Psi}\}dx. \tag{42}$$

The angular velocity and the rotation angles:

$$\{\omega_1(x)\} = \int_0^l [u_{e1me2me3m}]\{\dot{\Psi}\}dx \ , \tag{43}$$

$$\{\varphi(x)\} = \{\omega_1(x)\} \cdot \Delta t \ , \tag{44}$$

$$\{\varphi\} = [\alpha \quad \beta \quad \gamma]^T \ , \tag{45}$$

$$\{e\} = \{cos\, \varphi\}. \tag{46}$$

Further a new local stiffness matrix of the rod element is made using the generalized More’s formula.

3 RESULTS COMPARISON WITH FEM

The programs for the flat and spatial rod system were created for checking the developed mathematical model (Fig. 3). Separate program blocks for flat rod system are given in [2]. The calculation results were compared with the finite element method in ANSYS. The automatic time step feature has been disabled. The following initial data were accepted in the task:

- elastic modulus $E = 2{,}1 \cdot 10^{11}$ Pa;
- the tangent elastic modulus $E_{pl} = 2{,}1 \cdot 10^{10}$ Pa;
- Poisson’s ratio $\nu = 0{,}3$;
- yield strength $\sigma_s = 240$ MPa;
- square section width $a = 0{,}1$ m; $l = 1$ m;

- force $F = 1{\cdot}10^5\ N$.

As the developed mathematical model takes into account the stress distribution over the cross-section area, so in ANSYS the rod cross-section was divided into 20 sites by height and width. The number of finite elements on the rod is 100. For the discrete-analytical method calculation the rod was given 100 sites of integration. The nonlinear calculation was carried out in 20 steps. Calculation results are shown on Fig. 4, 5.

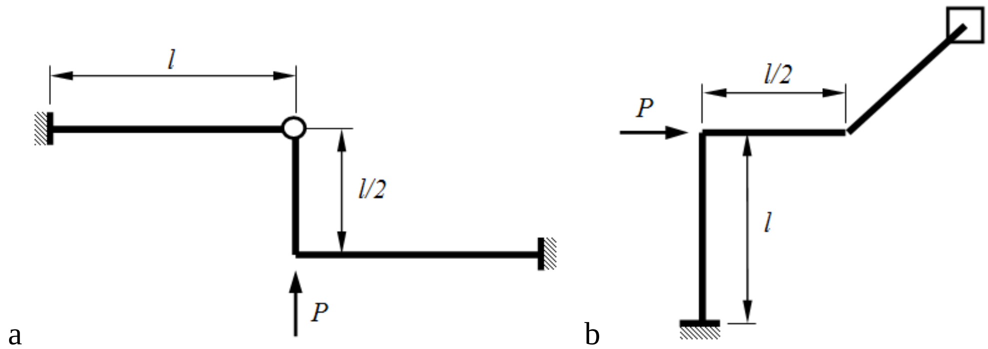

Figure 3: Analytical scheme of: a) a plane frame; b) a spatial frame

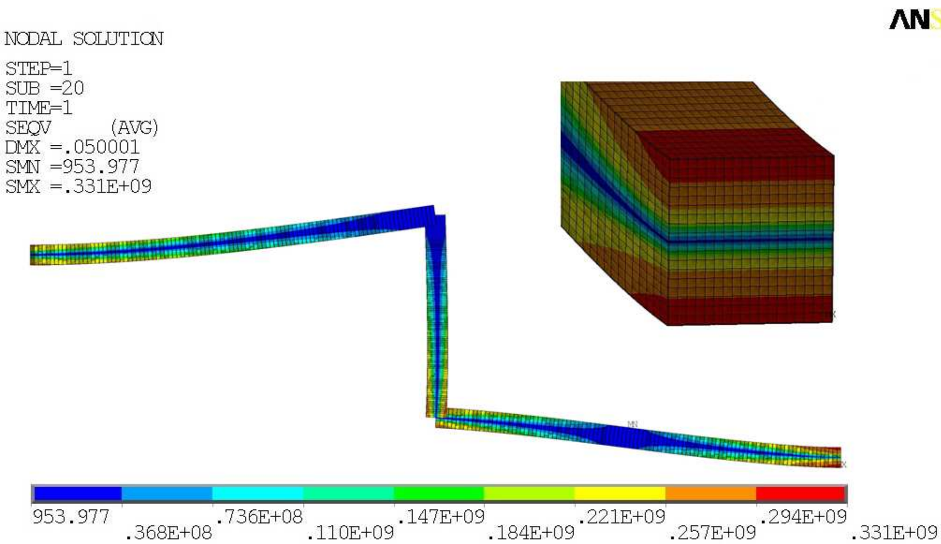

Figure 4: Stress distribution in a plane frame

Table 1: Plane frame (fisical nonlinearity)

Parameter	DANA	FEA (B188)	δ, %
σ, MPa	345	343	0,6
Δ, m	0,0553	0,0555	0,4
t, c	-	86	-

Table 2: Plane frame (geometrical nonlinearity)

Parameter	DANA	FEA (B188)	δ, %
σ, MPa	358	331	8
Δ, m	0,054	0,0500	8
t, c	-	90	-

Table 3: Spatial frame (fisical nonlinearity)

Parameter	DANA	FEA (B188)	δ, %
σ, MPa	408	406	0,5
Δ, m	0,094	0,09417	0,2
t, c	-	86	-

Table 4: Spatial frame (geometrical nonlinearity)

Parameter	DANA	FEA (B188)	δ, %
σ, MPa	440	415	6
Δ, m	0,105	0,09903	6
t, c	-	90	-

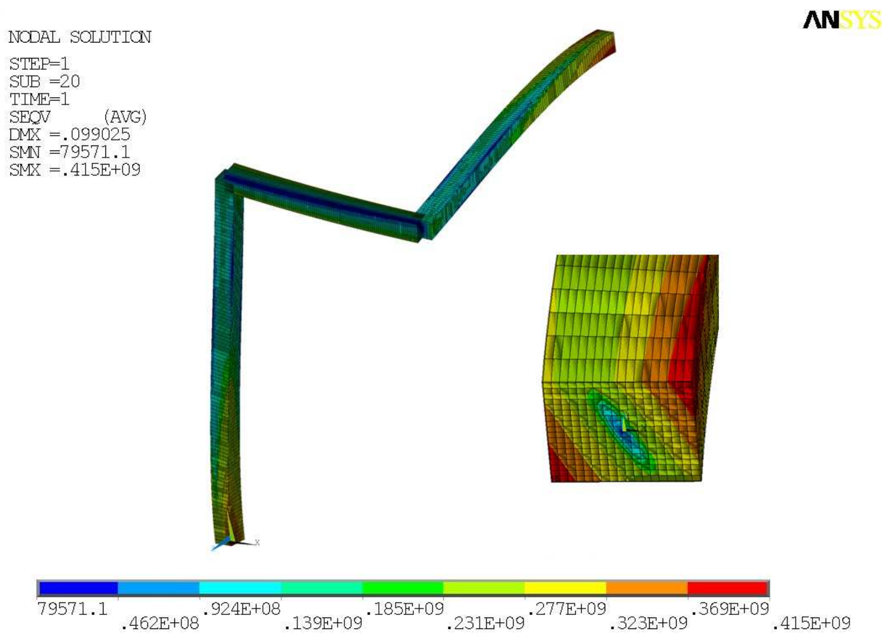

Figure 5: Stress distribution in a spatial frame

The comparison of obtained results with PC ANSYS reflected a satisfactory coincidence both at stress and at movements (Tab. 1-4). It is required 90 s of computer time to calculate by finite element method in case of one processor core loading (2,3 GHz).

If the section is not divided into elementary sites then the deviation from exact solution can achieve 15-20% as a result of incorrect transition from elastic to plastic zone of the diagram σ-ε by the cross-sectional area. During the rod dividing into 10 finite elements and using default setting of the ANSYS program the deviation from the exact solution was 15%.

It took less than 1 second for calculating via developed hybrid method. This fact is occurred because number of equations in the system is equal to the static uncertainty data, i.e. less than in the finite element method. In case of increasing the number of integration sites, the solution tends to be accurate without significant time costs.

4 CONCLUSIONS

The proposed discrete-analytical method allows to increase the accuracy in determining the stresses in framework and significantly reduce the time of elastoplastic calculation. The key features and main advantages of discrete-analytical nonlinear analysis in comparison with the finite element method are worth noting.

The key features:

- *a generalized More's formula with a tangential stiffness matrix* is proposed to determine the rod movements, which characterizes the stress state of all the rod section points;
- the use of the generalized More's formula with variable rigidity along the rod length allows to *exclude finite elements* along the rod;
- to calculate statically indeterminate systems a *hybrid FEM in the form of a displacement method* is used with analytical expressions for the rod jointly.

The main advantaged:

- the use of the tangential stiffness matrix allows *to increase the accuracy* of determination of *elastoplastic deformation* in the rods;
- the proposed method allows *to reduce the calculation time* at use the *explicit and implicit numerical integration schemes* implemented in powerful software packages based on FEM.
- by exclusion of finite elements the time of calculation of *differential equations motion systems by direct methods will be significantly reduced* considering the nonlinear structure behavior.

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