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ON AXIALLY SYMMETRIC SHELL PROBLEMS WITH REINFORCED JUNCTIONS

VIOLETTA KONOPIŃSKA-ZMYSŁOWSKA AND VICTOR A. EREMEYEV

Gdańsk University of Technology, Faculty of Civil and Environmental Engineering
ul. G. Narutowicza 11/12, 80-233 Gdańsk, Poland
violetta.konopinska@pg.edu.pl, victor.eremeyev@gmail.com, www.pg.edu.pl

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Abstract. Within the framework of the six-parameter nonlinear resultant shell theory we consider the axially symmetric deformations of a cylindrical shell linked to a circular plate. The reinforcement in the junction of the shell and the plate is taken into account. Within the theory the full kinematics is considered. Here we analyzed the compatibility conditions along the junction and their influence on the deformations and stressed state.

1 INTRODUCTION

Used in the engineering real shell structures usually consist of more than one regular shell element, therefore it is important to develop proper description of thin elements connections. Different theoretical, numerical and experimental approaches to modelling, analyses and design of the multi-fold shell structures with junctions is presented in [1]. Junctions between regular shell elements can be considered as rigid, simply-supported or deformable, they can also be treated as a reinforcement. In order to model junctions with reinforcements here we use the non-linear resultant shell theory presented by Libai and Simmonds [2], where the general, dynamically and kinematically exact six-field theory of regular shell was formulated with regard to a non-material surface as the shell base surface. Within the theory the kinematics of the shell is modelled using six degrees of freedom, so the motion of each point of the shell is described with three translations and three rotations. Here we consider multi-fold structures reinforced along junctions. Following [1,3,4] we present the compatibility conditions for junctions with reinforcements, which constitute the key-point of the problem. Other approaches are also known in the literature, see, e.g., [5–13].

As an example we consider static axial deformation of an elastic cylindrical shell connected with circular plate reinforced by an elastic ring along the junction. Restricting ourselves by small deformations we present the analytical solution of the problem. So it can be also used as a benchmark solution for more complex cases appeared in engineering.

2 NONLINEAR SHELL EQUILIBRIUM EQUATIONS

Following [1,3,4] we consider a shell as a three-dimensional (3D) solid thin body which in a reference (undeformed) placement is identified with a region B of the physical space $\mathcal{E}$ having the 3D vector space E as its translation space. Geometry of B is described in the normal coordinates (θ^α, ξ), $\alpha = 1, 2$, where $\xi = 0$ defines the regular shell base surface $M \subset B$, and $\xi \in [-h^-, h^+]$ is the distance from M, and $h = h^- + h^+$ is the shell thickness. In an inertial frame $(o, \mathbf{i}_k)$, where $o \in \mathcal{E}$ and $\mathbf{i}_k \in E$, $k = 1, 2, 3$, are orthonormal vectors, the position vector $\mathbf{x}$ of an arbitrary point $\mathrm{x} \in B$ is given by

$$\mathbf{x}(\theta^\alpha, \xi) = \boldsymbol{x}(\theta^\alpha) + \xi \boldsymbol{\eta}(\theta^\alpha), \tag{1}$$

where $\boldsymbol{x}(\theta^\alpha) = \mathbf{x}(\theta^\alpha, 0)$ is the position vector of M, $\boldsymbol{\eta} = \frac{1}{\sqrt{a}} \boldsymbol{x}_{,1} \times \boldsymbol{x}_{,2}$ is the unit normal vector of M, $a = \det(\boldsymbol{x}_{,\alpha} \cdot \boldsymbol{x}_{,\beta})$, and $(...)_{,\alpha} \equiv \frac{\partial}{\partial \theta^\alpha}(...)$.

In the deformed placement the shell is represented by the position vector $\boldsymbol{y} = \chi(\boldsymbol{x})$ of the deformed material base surface $\chi(M)$ with attached three directors $(\boldsymbol{d}_\alpha, \boldsymbol{d})$ such that

$$\boldsymbol{y} = \chi(\boldsymbol{x}) \equiv \boldsymbol{x} + \boldsymbol{u}, \quad \boldsymbol{d}_\alpha = \boldsymbol{Q}\boldsymbol{x}_{,\alpha}, \quad \boldsymbol{d} = \boldsymbol{Q}\boldsymbol{\eta}, \tag{2}$$

where χ is the deformation function, $\boldsymbol{u} \in E$ is the translation vector of M, and $\boldsymbol{Q} \in SO(3)$ is the proper orthogonal tensor, $\boldsymbol{Q}^T = \boldsymbol{Q}^{-1}$, $\det \boldsymbol{Q} = +1$. Defined on M tensor $\boldsymbol{Q}$ represents the work-averaged gross rotation of the shell cross-sections from their undeformed shapes.

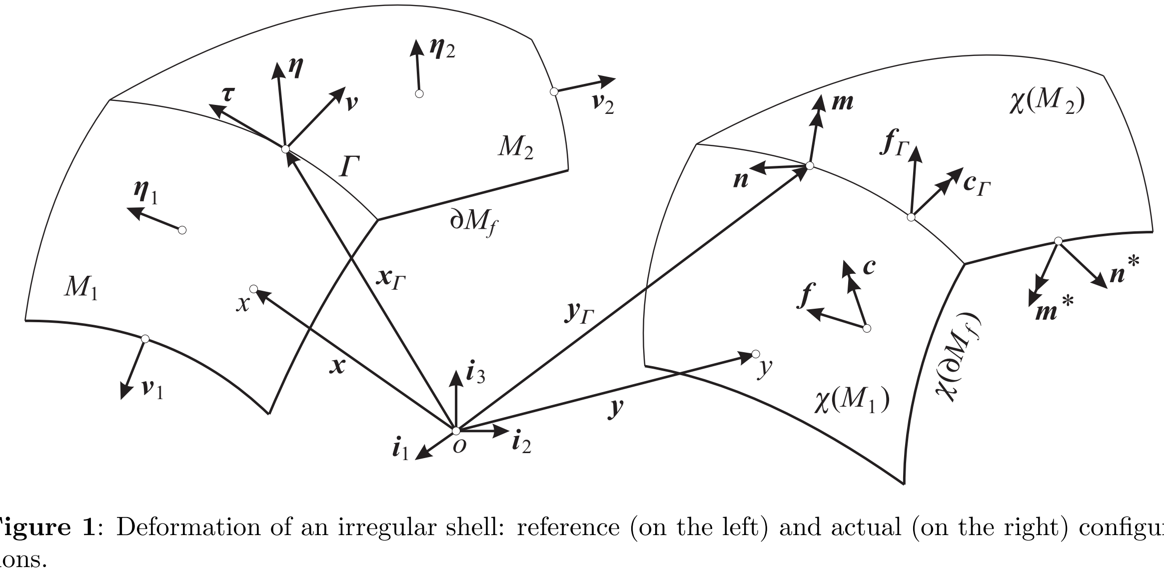

Figure 1: Deformation of an irregular shell: reference (on the left) and actual (on the right) configurations.

The exact resultant Lagrangian equilibrium conditions for the shell are derived by performing direct integration across the shell thickness of the 3D global equilibrium conditions of continuum mechanics, see for example [3]. Let $\boldsymbol{f}(\theta^\alpha)$, $\boldsymbol{c}(\theta^\alpha)$ be the resultant surface force and couple vector fields acting on $\chi(M)$, but measured per unit area of M, and let $\boldsymbol{n}^\star(s)$, $\boldsymbol{m}^\star(s)$ be the resultant 1D boundary force and couple vector fields acting along $\chi(\partial M_f)$, but measured per unit length of ∂M_f. Then the exact, resultant, local Lagrangian equilibrium conditions are [3]

$$\begin{aligned} \mathrm{Div}_s\boldsymbol{N}+\boldsymbol{f}=\boldsymbol{0}, \quad \mathrm{Div}_s\boldsymbol{M}+\mathrm{ax}(\boldsymbol{N}\boldsymbol{F}^T-\boldsymbol{F}\boldsymbol{N}^T)+\boldsymbol{c}=\boldsymbol{0} \quad \text{in } M\setminus\Gamma, \\ \boldsymbol{n}^\star-\boldsymbol{N}\boldsymbol{\nu}=\boldsymbol{0}, \quad \boldsymbol{m}^\star-\boldsymbol{M}\boldsymbol{\nu}=\boldsymbol{0} \quad \text{along } \partial M_f, \end{aligned} \tag{3}$$

where $(\boldsymbol{N},\boldsymbol{M})\in E\otimes T_xM$ are the surface tangential stress resultant and stress couple tensors of the first Piola-Kirchhoff type, following from the Cauchy theorem $\boldsymbol{n}_\nu=\boldsymbol{N}\boldsymbol{\nu}$ and $\boldsymbol{m}_\nu=\boldsymbol{M}\boldsymbol{\nu}$ of the resultant contact force $\boldsymbol{n}_\nu$ and couple $\boldsymbol{m}_\nu$ vectors, see Fig. 1, where $M=M_1\cup M_2$, Γ is the junction between two parts of the shell. $\boldsymbol{F}=\mathrm{Grad}_s\boldsymbol{y}$ is the surface deformation gradient, $\boldsymbol{F}\in E\otimes T_xM$, ax(...) is the axial vector associated with the skew tensor (...), $\boldsymbol{\nu}$ is the surface unit vector externally normal to ∂M, whereas Grad_s and Div_s are the surface gradient and divergence operators on M, respectively.

In the general theory of shells the strain and bending tensors $\boldsymbol{E}$ and $\boldsymbol{K}$ in the spatial representation are defined by the formulae

$$\boldsymbol{E}=\boldsymbol{\varepsilon}_\alpha\otimes\boldsymbol{a}^\alpha, \quad \boldsymbol{K}=\boldsymbol{\kappa}_\alpha\otimes\boldsymbol{a}^\alpha, \quad \boldsymbol{\varepsilon}_\alpha=\boldsymbol{y}_{,\alpha}-\boldsymbol{d}_\alpha, \quad \boldsymbol{\kappa}=\frac{1}{2}\boldsymbol{d}^i\times\boldsymbol{Q}_{,\alpha}\boldsymbol{Q}^T\boldsymbol{d}_i, \tag{4}$$

where $(\boldsymbol{a}^\alpha,\boldsymbol{\eta})$ and $(\boldsymbol{d}^i)$ are the base reciprocal to $(\boldsymbol{x}_{,\alpha},\boldsymbol{\eta})$ and $(\boldsymbol{d}_\alpha,\boldsymbol{d})$, respectively.

The referential shell stress and couple stress tensors as well as the referential shell strain measures are defined by the relations

$$\mathbf{N}=\boldsymbol{Q}^T\boldsymbol{N}, \quad \mathbf{M}=\boldsymbol{Q}^T\boldsymbol{M}, \quad \mathbf{E}=\boldsymbol{Q}^T\boldsymbol{E}, \quad \mathbf{K}=\boldsymbol{Q}^T\boldsymbol{K}. \tag{5}$$

For an isotropic elastic shell we consider the following form of the surface strain energy density W

$$\begin{aligned} W= &\ \alpha_1\mathrm{tr}^2\mathbf{E}_\parallel+\alpha_2\mathrm{tr}\mathbf{E}_\parallel^2+\alpha_3\mathrm{tr}(\mathbf{E}_\parallel^T\mathbf{E}_\parallel)+\alpha_4\boldsymbol{\eta}\cdot\mathbf{E}\mathbf{E}^T\boldsymbol{\eta} \\ + &\ \beta_1\mathrm{tr}^2\mathbf{K}_\parallel+\beta_2\mathrm{tr}\mathbf{K}_\parallel^2+\beta_3\mathrm{tr}(\mathbf{K}_\parallel^T\mathbf{K}_\parallel)+\beta_4\boldsymbol{\eta}\cdot\mathbf{K}\mathbf{K}^T\boldsymbol{\eta}, \end{aligned} \tag{6}$$

where

$$\mathbf{E}_\parallel=\mathbf{E}-\mathbf{E}^T\boldsymbol{\eta}, \quad \mathbf{K}_\parallel=\mathbf{K}-\mathbf{K}^T\boldsymbol{\eta},$$

denote the projections of $\mathbf{E}$ and $\mathbf{K}$ on the tangent space $T_xM\otimes T_xM$ to M at $x\in M$, and α_k, β_k are stiffness parameters, $k=1,2,3,4$. The stiffness parameters are given by

$$\begin{aligned} &\alpha_1=C\nu, \quad \alpha_2=0, \quad \alpha_3=C(1-\nu), \quad \alpha_4=\alpha_sC(1-\nu), \\ &\beta_1=D\nu, \quad \beta_2=0, \quad \beta_3=D(1-\nu), \quad \beta_4=\alpha_tD(1-\nu), \\ &C=\frac{Eh}{1-\nu^2}, \quad D=\frac{Eh^3}{12(1-\nu^2)}, \end{aligned} \tag{7}$$

where E and ν are the Young modulus and Poisson ratio of the bulk material, respectively, α_s and α_t are dimensionless coefficients, which play a role of shear correction factors, and h is the shell thickness.

W generates the following constitutive equations for $\mathbf{N}$ and $\mathbf{M}$

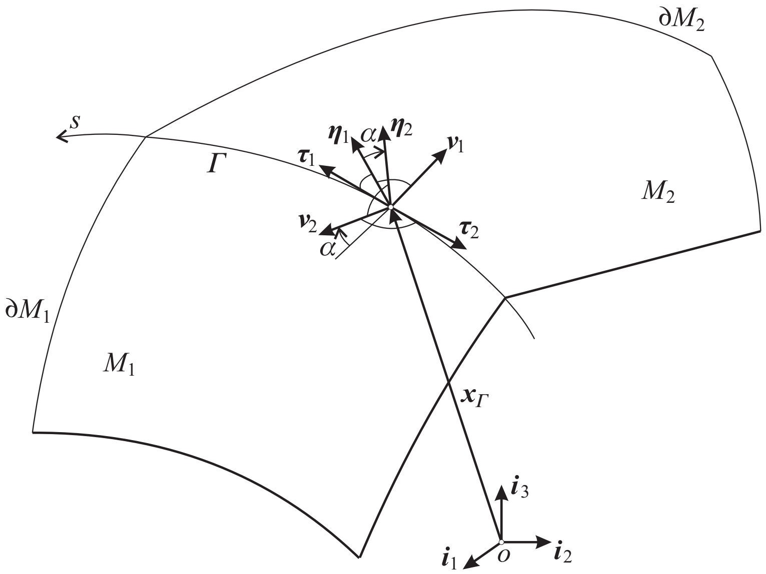

Figure 2: Geometry in the vicinity of the junction

$$\mathbf{N} = \alpha_1 \boldsymbol{A}\mathrm{tr}\mathbf{E}_{\|} + \alpha_2 \mathbf{E}_{\|}^T + \alpha_3 \mathbf{E}_{\|} + \alpha_4 \boldsymbol{\eta} \otimes \mathbf{E}^T \boldsymbol{\eta}, \qquad (8)$$
$$\mathbf{M} = \beta_1 \boldsymbol{A}\mathrm{tr}\mathbf{K}_{\|} + \beta_2 \mathbf{K}_{\|}^T + \beta_3 \mathbf{K}_{\|} + \beta_4 \boldsymbol{\eta} \otimes \mathbf{K}^T \boldsymbol{\eta},$$

where $\boldsymbol{A} = \mathbf{1} - \boldsymbol{\eta} \otimes \boldsymbol{\eta}$, $\boldsymbol{\eta} \cdot \boldsymbol{A}\boldsymbol{\eta} = 0$, while $\mathbf{1} \in E \otimes E$ and $\boldsymbol{A} \in T_x M \otimes T_x M$ are metric tensors of the 3D space and of the undeformed base surface, respectively.

Let us consider the compatibility conditions along the junctions, see two folds connected along curve Γ in Fig. 2. We treat the junction as a elastic reinforcement with additional linear strain energy and stress measures $\boldsymbol{n}$ and $\boldsymbol{m}$. We model the reinforcement using the elastic Cosserat curve, that is 1D elastic continuum with additional constitutive relations, see, e.g., [4] and the reference therein. The compatibility conditions along Γ are given by

$$\boldsymbol{n}' + [\![\, \boldsymbol{N}\boldsymbol{\nu} \,]\!] = \mathbf{0}\,, \quad \boldsymbol{m}' + \boldsymbol{y}_\Gamma' \times \boldsymbol{n} + [\![\, \boldsymbol{M}\boldsymbol{\nu} \,]\!] = \mathbf{0}\,. \qquad (9)$$

Here $\boldsymbol{y}_\Gamma = \chi(\boldsymbol{x}_\Gamma)$ is the position of Γ in the actual configuration, $\boldsymbol{n}$ and $\boldsymbol{m}$ are the stress resultant and stress couple vectors defined on Γ, the double square brackets denote a discontinuity jump across Γ, the and the prime stands for the derivative with respect to the arc-length s along Γ. Eqs. (9) can be derived from the virtual work principle, see [4] for details. Let us note that the Cosserat curve model is kinematically consistent with the six-parameter shell theory. In particular, one can describe the stretching, bending and torsion of the reinforcement together with the deformations of the shell. The similar model of reinforcements was also used in [7].

For small deformations, we simplify expressions of the strain measures (4) into the forms

$$\boldsymbol{\varepsilon}_\alpha = \boldsymbol{u}_{,\alpha} - \boldsymbol{\varphi} \times \boldsymbol{x}_{,\alpha}, \quad \boldsymbol{\kappa}_\alpha = \boldsymbol{\varphi}_{,\alpha}, \qquad (10)$$

where $\boldsymbol{\varphi}$ is the infinitesimal rotation vector such that $\boldsymbol{Q} \approx \mathbf{1} - \boldsymbol{\varphi} \times \mathbf{1}$ if $\|\boldsymbol{\varphi}\| \ll 1$, and $\mathbf{1}$ is the 3D identity tensor. In such a case we approximately have $\boldsymbol{N} \cong \mathbf{N}$, $\boldsymbol{M} \cong \mathbf{M}$, $\boldsymbol{E} \cong \mathbf{E}$, $\boldsymbol{K} \cong \mathbf{K}$.

In what follows we consider small axially symmetric deformations of a cylindrical shell connected to a plate, see Fig.3. The typical junction and the related free body diagram are shown in Fig. 4.

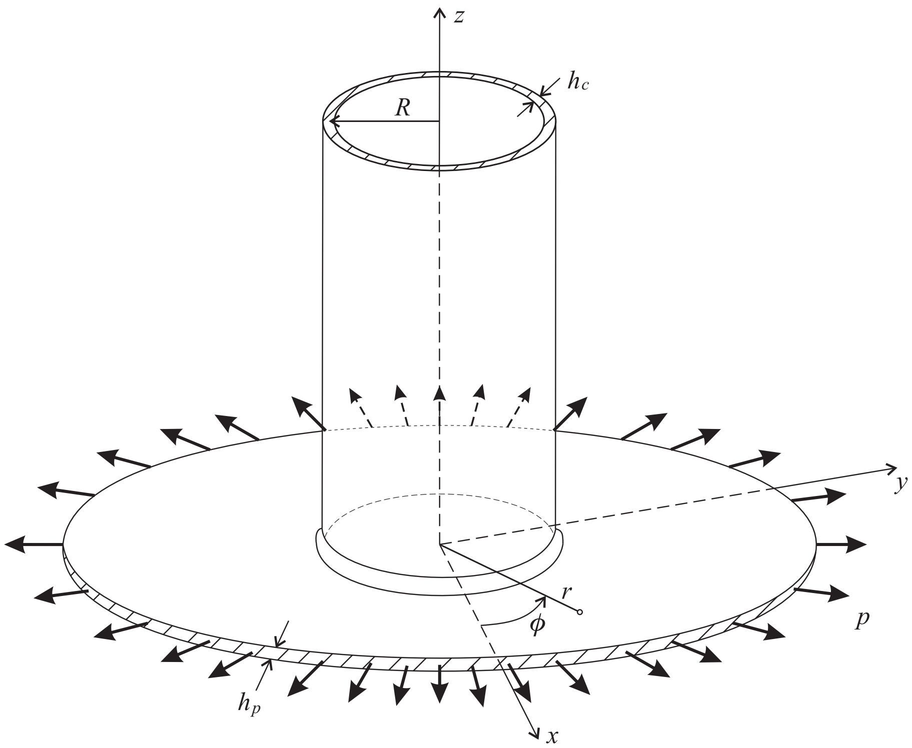

Figure 3: Connection: cylindrical shell – circular plate

3 AXIALLY SYMMETRIC DEFORMATIONS OF A CYLINDRICAL SHELL

Let us consider a thin circular cylindrical elastic shell of length L and of radius R. An axisymmetric loading acting on shell structure produces an axisymmetric deformation state of the form

$$\boldsymbol{u} = u(z)\mathbf{e}_z + w(z)\mathbf{e}_r, \quad \boldsymbol{\varphi} = \varphi(z)\mathbf{e}_\phi, \tag{11}$$

where $\mathbf{e}_r = \cos\phi\mathbf{i}_1 + \sin\phi\mathbf{i}_2$, $\mathbf{e}_\phi = -\sin\phi\mathbf{i}_1 + \cos\phi\mathbf{i}_2$, $\mathbf{e}_z = \mathbf{i}_3$ are the unit base vectors of the cylindrical system of coordinates. By applying (11) the linearized strain measures take the form

$$\mathbf{E} = \mathrm{Grad}_s\boldsymbol{u} - \boldsymbol{\varphi} \times \boldsymbol{A} = u'\mathbf{e}_z \otimes \mathbf{e}_z + (w' - \varphi)\mathbf{e}_r \otimes \mathbf{e}_z + \frac{w}{R}\mathbf{e}_\phi \otimes \mathbf{e}_\phi, \tag{12}$$
$$\mathbf{K} = \mathrm{Grad}_s\boldsymbol{\varphi} = \varphi'\mathbf{e}_\phi \otimes \mathbf{e}_z - \frac{\varphi}{R}\mathbf{e}_r \otimes \mathbf{e}_\phi,$$

where $(...)' = \frac{\partial}{\partial z}(...)$.

The surface stress measures $\mathbf{N}$ and $\mathbf{M}$ of the axisymmetric stress state are given by

$$\begin{aligned} \mathbf{N} &= N_{zz}\mathbf{e}_z \otimes \mathbf{e}_z + N_{\phi\phi}\mathbf{e}_\phi \otimes \mathbf{e}_\phi + N_{rz}\mathbf{e}_r \otimes \mathbf{e}_z, \\ \mathbf{M} &= M_{\phi z}\mathbf{e}_\phi \otimes \mathbf{e}_z + M_{z\phi}\mathbf{e}_z \otimes \mathbf{e}_\phi + M_{r\phi}\mathbf{e}_r \otimes \mathbf{e}_\phi. \end{aligned} \tag{13}$$

The equilibrium conditions (3) transform into the simple form

$$N'_{zz} = 0, \quad N'_{rz} - \frac{N_{\phi\phi}}{R} = 0, \quad M'_{\phi z} + \frac{M_{r\phi}}{R} + N_{rz} = 0, \tag{14}$$

From $(14)_1$ it immediately follows that $N_{zz} = P \equiv \text{const}$. The latter equations take the form of the following five ODEs

$$\begin{aligned} u' &= \frac{P}{C} - \nu\frac{w}{R}, \quad w' = \frac{N_{rz}}{\alpha_4} + \varphi, \quad \varphi' = \frac{M_{\phi z}}{D(1-\nu)}, \\ N'_{rz} &= \frac{N_{\phi\phi}}{R}, \quad N_{\phi\phi} = \nu P + C(1-\nu^2)\frac{w}{R}, \\ M'_{\phi z} &= -\frac{M_{r\phi}}{R} - N_{rz}, \quad M_{r\phi} = -\beta_4\frac{\varphi}{R}. \end{aligned} \tag{15}$$

For brevity we omitted here the general form of the solution of (15), let us only underline that here we have five integration constants, in general.

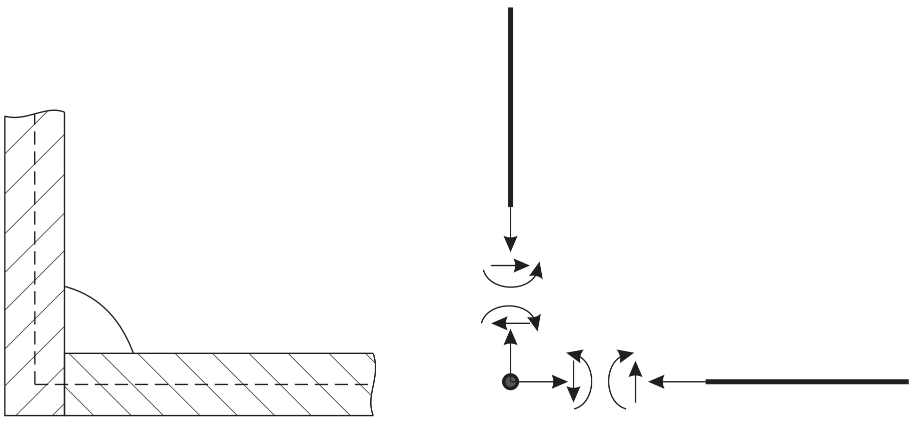

Figure 4: Junction and its free body diagram

4 CIRCULAR PLATE UNDER AXISYMMETRIC LOAD

Let us consider the axisymmetric deformation of elastic circular plate under the action of tensile forces p, see Fig. 3. We again assume that strains and deformations are small.

The axisymmetric deformation of the plate is described by

$$\begin{aligned} \mathbf{E} &= \mathrm{Grad}_s\boldsymbol{u} - \boldsymbol{\varphi} \times \mathbf{1} = u'\mathbf{e}_r \otimes \mathbf{e}_r + (w' - \varphi)\mathbf{e}_z \otimes \mathbf{e}_r + \frac{u}{r}\mathbf{e}_\phi \otimes \mathbf{e}_\phi, \\ \mathbf{K} &= \mathrm{Grad}_s\boldsymbol{\varphi} = \varphi'\mathbf{e}_\phi \otimes \mathbf{e}_r - \frac{\varphi}{r}\mathbf{e}_r \otimes \mathbf{e}_\phi, \end{aligned} \tag{16}$$

where $(...)'$ denotes now the derivative with respect to r. The stress measures $\mathbf{N}$ and $\mathbf{M}$ take the form

$$\mathbf{N} = N_{rr}\mathbf{e}_r \otimes \mathbf{e}_r + N_{\phi\phi}\mathbf{e}_\phi \otimes \mathbf{e}_\phi + N_{zr}\mathbf{e}_z \otimes \mathbf{e}_r, \qquad (17)$$
$$\mathbf{M} = M_{\phi r}\mathbf{e}_\phi \otimes \mathbf{e}_r + M_{r\phi}\mathbf{e}_r \otimes \mathbf{e}_\phi.$$

From (8) and (16) it follows that

$$N_{rr} = \alpha_1(u' + \frac{u}{r}) + \alpha_3 u'\,, \quad N_{\phi\phi} = \alpha_1(u' + \frac{u}{r}) + \alpha_3\frac{u}{r}\,, \quad N_{zr} = \alpha_4(w' - \varphi)\,, \qquad (18)$$
$$M_{\phi r} = \beta_3\varphi', \quad M_{r\phi} = -\beta_3\frac{\varphi}{r}.$$

Equilibrium equations (3) reduce here to three ordinary differential equations

$$N'_{rr} + \frac{1}{r}(N_{rr} - N_{\phi\phi}) = 0, \quad N'_{zr} + \frac{1}{r}N_{zr} = 0, \qquad (19)$$
$$M'_{\phi r} + \frac{1}{r}(M_{\phi r} + M_{r\phi}) = 0.$$

where we assumed that $\boldsymbol{f} = \boldsymbol{c} = \mathbf{0}$. As a result, N_{zr} is given by

$$N_{zr} = \frac{c_1}{r}, \qquad (20)$$

where c_1 is the integration constant.

Substituting (18) into (19) we obtain three 2nd-order relations for u, w and φ

$$w = w_0 + \frac{c_1 r^2}{2} + c_2 \ln r, \quad u = d_1 r + \frac{d_2}{r}, \quad \varphi = c_1 r + \frac{c_2}{r}, \qquad (21)$$

where c_1, c_2, d_1, d_2 and $w_0$are integration constant.

The boundary conditions for the plate are given by the relations

$$N_{rr} = p, \quad M_{\phi r} = 0 \qquad (22)$$

at the external boundary of the plate.

In the considered case the reinforcement coincides with an elastic circle which undergoes stretching and torsion. Combining solutions for the shell and plate one can obtain the system of linear algebraic equations for the integration constants. Note that the torsion plays here an important role and leads to the bending of the both plates and shells. The detailed discussion of the solutions for various boundary conditions preserving axially symmetric deformations will be presented during the conference.

5 CONCLUSIONS

In the paper we discuss the benchmark solution for an elastic cylindrical shell connected to a circular plate considering also an reinforcement along the junction. Using the six-parameter shell theory and the Cosserat curve model of reinforcement we obtain an analytical solution for the problem for various boundary conditions. We discuss in brief the peculiarities of the problem. In particular, within the considered shell theory we have a bending due to stretching of the plate which was not not present in solution given in [7].

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