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VALIDATION OF A LINEARLY ELASTIC PERIDYNAMIC MATERIAL

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Abstract. Peridynamics is a nonlocal theory that extends the classical continuum theory by considering collective motion of all the material within a δ-neighborhood of any point of a peridynamic body. It considers the interaction of material points due to forces acting at a finite distance smaller than δ, which is called the peridynamic horizon. A relation between interaction force and relative displacement between particles was proposed in previous work for an isotropic linear elastic peridynamic material. The relation is derived from a free energy function that depends quadratically on measures of strain that are analogous to the measures of strain of the classical linear theory. The energy function contains four peridynamic material constants; three of which were determined in previous work by using both convergence results of the peridynamic theory to the classical linear elasticity theory and a correspondence argument between the proposed free energy function and the strain energy density function from the classical linear elasticity theory. We have also shown an expression for the fourth material constant, which was obtained from the correspondence argument by evaluating both the peridynamic free energy and the strain energy at a specific point of a beam bent by terminal couples. In this work we show that this expression is valid regardless of the point chosen inside the beam. Also, we have considered two additional experiments to verify the validity of the expressions obtained for all the peridynamic constants, results of which will be presented at the conference. This work is of interest in all areas of continuum mechanics, such as in fracture mechanics, where surfaces of discontinuities exist, or, may appear as a result of deformation.

1 INTRODUCTION

In Peridynamics interaction forces between material points acting at a finite distance smaller than a peridynamic horizon δ are related to relative displacements, and the balance of linear momentum is formulated as an integral equation that remains valid across a surface of discontinuity, or, a continuum in which discontinuities may appear as a result of deformation. Away from these places, where the deformation is smooth, peridynamics yields the same governing equations of the classical continuum theory in the limit of vanishing distances between material points.

A relation between interaction force and relative displacement between particles is presented by Aguiar and Fosdick in [1] for an isotropic linear elastic peridynamic material. The relation is derived from a free energy function that depends quadratically on measures of strain that are analogous to the measures of strain of the classical linear theory. The free energy function contains four peridynamic material constants. Using homogeneous deformations and expressions presented by Silling in [4] that relate the elasticity tensor from the classical linear theory to its counterpart in peridynamics, called the modulus state, Aguiar and Fosdick in [1] derive two relations between the two Lamé constants and three peridynamic constants, leaving, therefore, one constant in these two relations as arbitrary.

To determine the arbitrary constant, Aguiar in [2] introduces a decomposition of the relative displacement in terms of radial and non-radial components. If the radial component is zero, the free energy function reduces to an integral expression that multiplies the arbitrary constant. This result is then used in a correspondence argument between the free energy function evaluated at any material point and a weighted average of the strain energy function of classical linear theory in a δ-neighborhood of this point, yielding a general expression for the determination of the arbitrary constant. To generate the zero radial component, this author considers the torsion of a circular shaft in equilibrium without body force. The resulting expression together with the previous two expressions yield the three peridynamic constants referred to above.

The procedure described above to determine the third arbitrary constant is used by Seitenfuss, Aguiar, and Pereira in [3] to determine the fourth peridynamic material constant. For this, the authors consider the experiment of a cylindrical beam bent by terminal couples and evaluate both the peridynamic free energy function and the weighted average of the strain energy function at the origin of the coordinate system, which is located at the center of one end of the beam.

This work consists of verifying the validity of the expressions obtained for the peridynamic material constants. For this, we consider again the experiment of a cylindrical beam bent by terminal couples, but do not restrict the previous evaluations to the origin of the coordinate system. We also consider other experiments in mechanics, such as bending of a beam by terminal load and anti-plane shear of a circular cylinder, which will be discussed at the conference. Analytical and numerical results indicate that the expressions for the four peridynamic constants are independent of the experiment chosen.

In summary, in Section 2 we present preliminary results concerning the kinematics of small deformations, which is used in the presentation of the free energy function of an isotropic simple elastic material containing four peridynamic coefficients. We then review the procedure to obtain three of these coefficients in Section 2.3 and the fourth one in Section 2.4. Recall from above that, for the fourth coefficient, we calculate both the peridynamic free energy function and the weighted average of the classical strain energy density at the center of an end of a beam bent by terminal couples. In Section 3 we show that the expression for the fourth constant does not depend on the particular choice of a material point inside this beam. In Section 4 we present concluding remarks.

2 PRELIMINARY RESULTS

2.1 Kinematics of small deformation

Let $\mathcal{B} \in \mathbb{E}^3$ be the undistorted reference configuration of a body and $\mathbf{x} \in \mathcal{B}$ be a material point of $\mathcal{B}$. Let also $N_\delta(\mathbf{x}_0) \subset \mathcal{B}$ be a neighborhood of any point $\mathbf{x}_0 \in \mathcal{B}$. Here, $N_\delta(\mathbf{x}_0)$ is a sphere of radius δ centered at $\mathbf{x}_0$. For $\mathbf{x}_0 \in N_\delta(\mathbf{x}_0)$, the vector $\boldsymbol{\xi} := \mathbf{x} - \mathbf{x}_0$ is called a bond of $\mathbf{x}$ to $\mathbf{x}_0$ and $\mathcal{H}_\delta(\mathbf{x}_0)$ is the collection of all bonds to $\mathbf{x}_0$.

A peridynamic state at $(\mathbf{x}_0, t)$ of order m is a function $\underline{\mathbf{A}}(\mathbf{x}_0, t)\langle\cdot\rangle : \mathcal{H}_\delta(\mathbf{x}_0) \to \mathcal{L}_m$, where $\mathcal{L}_m$ is the set of all tensors of order m. Thus, the image of a bond $\xi \in \mathcal{H}_\delta(\mathbf{x}_0)$ for the state $\underline{\mathbf{A}}(\mathbf{x}_0, t)\langle\cdot\rangle$ is the tensor of order m, $\underline{\mathbf{A}}(\mathbf{x}_0, t)\langle\boldsymbol{\xi}\rangle$. We denote by $\mathcal{A}_m$ the set of all states at $(\mathbf{x}_0, t)$ of order m. The dependency between two states $\underline{\mathbf{A}}(\mathbf{x}_0, t)\langle\cdot\rangle : \mathcal{H}_\delta(\mathbf{x}_0) \to \mathcal{L}_m$ and $\underline{\mathbf{u}}(\mathbf{x}_0, t)\langle\cdot\rangle : \mathcal{H}_\delta(\mathbf{x}_0) \to \mathcal{L}_p$ is denoted by $\underline{\mathbf{A}}(\mathbf{x}_0, t)\langle\boldsymbol{\xi}\rangle = \widehat{\underline{\mathbf{A}}}(\mathbf{x}_0, t)[\underline{\mathbf{u}}]\langle\boldsymbol{\xi}\rangle$. For notational convenience, we shall not exhibit the dependence on the time variable t and, when the meaning is clear, may also omit the dependence on the particle $\mathbf{x}_0$.

The difference deformation state $\underline{\boldsymbol{\chi}} \in \mathcal{A}_1$ at $\mathbf{x}_0 \in \mathcal{B}$ is defined through

$$\underline{\boldsymbol{\chi}}\langle\boldsymbol{\xi}\rangle := (\boldsymbol{\chi}(\mathbf{x}) - \boldsymbol{\chi}(\mathbf{x}_0))\,|_{\mathbf{x}=\mathbf{x}_0+\boldsymbol{\xi}}\,,$$

where $\boldsymbol{\chi}(\mathbf{x})$ is the motion of particle $\mathbf{x}$ at time t. A similar definition holds for the difference displacement state $\underline{\mathbf{u}} \in \mathcal{A}_1$ at $\mathbf{x}_0 \in \mathcal{B}$, which is given by

$$\underline{\mathbf{u}}\langle\boldsymbol{\xi}\rangle := (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0))\,|_{\mathbf{x}=\mathbf{x}_0+\boldsymbol{\xi}}\,,$$

where $\mathbf{u}(\mathbf{x})$ is the displacement of particle $\mathbf{x}$ at time t. With $\underline{\mathbf{x}} \in \mathcal{A}_1$ being the reference position vector state at $\mathbf{x}_0 \in \mathcal{B}$, so that $\underline{\mathbf{x}}\langle\boldsymbol{\xi}\rangle \equiv \boldsymbol{\xi} = \mathbf{x} - \mathbf{x}_0$, we may write $\underline{\boldsymbol{\xi}} = \underline{\mathbf{u}} + \underline{\mathbf{x}}$. The difference deformation and displacement quotient states at $\mathbf{x}_0 \in \mathcal{B}$ are then defined by

$$\underline{\mathbf{f}} := \frac{\underline{\boldsymbol{\chi}}}{|\underline{\mathbf{x}}|} = \underline{\mathbf{h}} + \underline{\mathbf{e}}, \qquad \underline{\mathbf{h}} := \frac{\underline{\mathbf{u}}}{|\underline{\mathbf{x}}|}, \tag{1}$$

respectively, where $\underline{\mathbf{e}} := \underline{\mathbf{x}}/\,|\underline{\mathbf{x}}|$ and $|\underline{\mathbf{A}}|$ is the magnitude state of $\underline{\mathbf{A}}$, defined through $|\underline{\mathbf{A}}|\,\langle\boldsymbol{\xi}\rangle := \sqrt{\underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{A}}\langle\boldsymbol{\xi}\rangle}$, with " $\cdot$ " being the scalar product in $\mathbb{E}^3$.

The difference displacement quotient state at $\mathbf{x}_0 \in \mathcal{B}$, defined in (1), can be decomposed as

$$\underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle = \underline{\varphi}\langle\boldsymbol{\xi}\rangle\underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle, \tag{2}$$

where $\underline{\varphi}$ is a scalar state that yields the radial component of $\underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle$ and $\underline{\mathbf{h}}_d$ is a vector state that satisfies $\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle \cdot \underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle = 0$.

2.2 The peridynamic material

The peridynamic equation of motion is given by ([1])

$$\rho(\mathbf{x}_0)\ddot{\mathbf{u}}(\mathbf{x}_0) = \int_{N_\delta} \{\underline{\mathbf{L}}(\mathbf{x}_0)\langle\mathbf{x} - \mathbf{x}_0\rangle - \underline{\mathbf{L}}(\mathbf{x})\langle\mathbf{x}_0 - \mathbf{x}\rangle\}dv_x + \mathbf{b}(\mathbf{x}_0), \tag{3}$$

where ρ is the mass density, $\mathbf{u}$ is the displacement field, $\mathbf{b}$ is a prescribed body force density, and $\underline{\mathbf{L}}(\mathbf{x}_0)\langle\cdot\rangle$ is the force vector state evaluated on bonds at $\mathbf{x}_0$. Equation (3) is the counterpart in peridynamics of the differential equation of balance of linear momentum from the classical theory. For a simple elastic material near its natural state,

$$\underline{\mathbf{L}}(\mathbf{x}_0) \equiv \widehat{\underline{\mathbf{L}}}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \frac{\delta_{\underline{\mathbf{h}}}\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}]}{|\underline{\mathbf{x}}|}, \tag{4}$$

where $\delta_{\underline{\mathbf{h}}}$ is the Fréchet derivative with respect to $\underline{\mathbf{h}}$ and $\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}]$ is a free energy function.

In [2] Aguiar uses the decomposition in (2) to present an alternative form for the quadratic free energy function of a simple elastic material proposed in [1]. It is given by

$$\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \widehat{W}_{\mathbf{x}_0}[\underline{\varphi}\,\underline{\mathbf{e}}] + \widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}_d] + \frac{\widehat{\alpha}_{13}}{2}\int_{N_\delta}\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle\cdot\int_{N_\delta}\frac{\omega(|\boldsymbol{\xi}|,|\boldsymbol{\eta}|)}{\sin\alpha}(\underline{\varphi}\langle\boldsymbol{\xi}\rangle+\underline{\varphi}\langle\boldsymbol{\eta}\rangle)\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle dv_\eta dv_\xi, \tag{5}$$

where $\omega(\cdot,\cdot)$ is a given symmetric weighting function,

$$\widehat{W}_{\mathbf{x}_0}[\underline{\varphi}\,\underline{\mathbf{e}}] = \frac{1}{2}\int_{N_\delta}\underline{\varphi}\langle\boldsymbol{\xi}\rangle\cdot\int_{N_\delta}\omega(|\boldsymbol{\xi}|,|\boldsymbol{\eta}|)[\widehat{\alpha}_{11}\underline{\varphi}\langle\boldsymbol{\xi}\rangle + 2\alpha_{12}\underline{\varphi}\langle\boldsymbol{\eta}\rangle]dv_\eta dv_\xi, \tag{6}$$

$$\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}_d] = \frac{\alpha_{33}}{4}\int_{N_\delta}\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle\cdot\int_{N_\delta}\frac{\omega(|\boldsymbol{\xi}|,|\boldsymbol{\eta}|)}{(\sin\alpha)^2}[\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle\cdot\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle + \underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle\cdot\underline{\mathbf{h}}_d\langle\boldsymbol{\eta}\rangle]\underline{\mathbf{e}}\langle\boldsymbol{\eta}\rangle dv_\eta dv_\xi, \tag{7}$$

and $\widehat{\alpha}_{11}$, α_{12}, $\widehat{\alpha}_{13}$, and α_{33} are elastic peridynamic constants.

2.3 Determination of three peridynamic constants

Using convergence results presented in [5], it is shown in [1] that

$$\widehat{W}^L_{\mathbf{x}_0}[\mathbf{E}] = \widehat{W}_{\mathbf{x}_0}[\mathbf{H}_0\underline{\mathbf{e}}], \tag{8}$$

where

$$\widehat{W}^L_{\mathbf{x}_0}[\mathbf{E}] = \frac{1}{2}[\lambda(tr\mathbf{E})^2 + 2\mu\mathbf{E}\cdot\mathbf{E}] \tag{9}$$

is the strain energy function of an isotropic classical linear elastic material and $\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}]$ is given by (5) together with both (6) and (7). In both (8) and (9), $\mathbf{H}_0$ is an infinitesimal displacement gradient, $\mathbf{E} := (\mathbf{H}_0 + \mathbf{H}_0^T)/2$ is the infinitesimal strain tensor, and both λ and μ are the Lamé constants. The authors obtain two relations between the peridynamic material constants $\widehat{\alpha}_{11}$, α_{12}, α_{33} and classical elasticity constants, which are given by

$$2\widehat{\alpha}_{11} + \alpha_{33} = \frac{15E}{16(1+\nu)\omega_\delta}, \quad \widehat{\alpha}_{11} + 2\alpha_{12} = \frac{3E}{16(1-2\nu)\omega_\delta}, \tag{10}$$

where $\nu = \lambda/[2\,(\lambda+\mu)]$ is the Poisson's ratio, $E = \mu\,(3\,\lambda + 2\,\mu)/(\lambda+\mu)$ is the Young's modulus, and $\omega_\delta \equiv \pi^2\int_0^\delta\int_0^\delta \omega(\check{\rho},\hat{\rho})\check{\rho}^2\hat{\rho}^2 d\hat{\rho}d\check{\rho}$.

To obtain a third relation, it is assumed in [2] that both the multiplicative decomposition

$$\omega(|\boldsymbol{\xi}|,|\boldsymbol{\eta}|) = \widetilde{\omega}(|\boldsymbol{\xi}|)\widetilde{\omega}(|\boldsymbol{\eta}|)|\boldsymbol{\xi}|^2|\boldsymbol{\eta}|^2\,, \tag{11}$$

where $\widetilde{\omega} : \mathbb{R} \to \mathbb{R}$ is a known weighting function, and the correspondence relation

$$\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \overline{W}^L_{\mathbf{x}_0}[\underline{\mathbf{h}}] := \frac{1}{m} \int_{N_\delta} \widetilde{\omega}(|\boldsymbol{\xi}|)\, |\boldsymbol{\xi}|^2\, \widehat{W}^L_{\mathbf{x}_0}[\widehat{\mathbf{E}}[\underline{\mathbf{h}}]] dv_\xi, \tag{12}$$

hold. In (12), $\widehat{\mathbf{E}}[\underline{\mathbf{h}}]$ is the infinitesimal strain tensor obtained from the vector state $\underline{\mathbf{h}}$ and

$$m := \int_{N_\delta} \widetilde{\omega}(|\boldsymbol{\xi}|)|\boldsymbol{\xi}|^2 dv_{\boldsymbol{\xi}}\, . \tag{13}$$

Observe from (12) with $\widehat{\mathbf{E}}[\underline{\mathbf{h}}]$ constant that $\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \widehat{W}^L_{\mathbf{x}_0}[\widehat{\mathbf{E}}[\underline{\mathbf{h}}]]$. Therefore, the relation (8) with $\omega(\cdot, \cdot)$ given by (11) is a particular case of (12).

Considering the infinitesimal deformation of a homogeneous, isotropic, and linearly elastic circular shaft under uniform torsion and using the correspondence relation (12), it is shown in [2] that

$$\alpha_{33} = \frac{20\mu}{m^2}. \tag{14}$$

Replacing (14) and the expressions $\mu = E/(2(1+\nu))$ and $\kappa = E/(3(1-2\nu))$ into (10), the other two contants can also be determined, being given by

$$\widehat{\alpha}_{11} = \frac{5\mu}{m^2}, \quad \alpha_{12} = \frac{1}{2m^2}(9\kappa - 5\mu)\ . \tag{15}$$

2.4 Determination of fourth peridynamic constant, $\hat{\alpha}_{13}$

In this section and in Section 3 we use a fixed orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ associated to the Cartesian coordinates (ξ_1, ξ_2, ξ_3) with origin at the centroid of the left end of a prismatic bar, which is aligned with the ξ_3-direction. The bar is homogeneous, isotropic and linearly elastic. To perform numerical integrations, we use the software MATHEMATICA 9 © with a global adaptive strategy.

We now review the procedure used in [3] to determine the fourth peridynamic constant, $\hat{\alpha}_{13}$. This constant represents nonlocal effects of the peridynamic material and can not be determined from the approach leading to the expressions in (10). To determine $\hat{\alpha}_{13}$, we consider a simple experiment in mechanics that provides a deformation field for which both radial and non-radial components of $\underline{\mathbf{h}}$ in (2) do not vanish.

The experiment consists of a beam bent by terminal couples in equilibrium with no body force, as illustrated in Fig. 1. The lateral surface of the beam is free of traction. In classical linear elasticity, the corresponding displacement field is given by (Sokolnikoff, 1956)

$$\mathbf{u}(\xi_1, \xi_2, \xi_3) = \frac{M}{2EI}[({\xi_3}^2 + \nu{\xi_1}^2 - \nu{\xi_2}^2)\mathbf{e}_1 + 2\nu\xi_1\xi_2\mathbf{e}_2 - 2\xi_1\xi_3\mathbf{e}_3], \tag{16}$$

where $M > 0$ is the magnitude of the bending moment, I is the moment of inertia with respect to the ξ_2-direction, and we recall from (10) that ν is the Poisson's ratio and E is the Young's modulus. The non-zero components of the corresponding infinitesimal strain tensor $\mathbf{E}$ are given by

$$\epsilon_{11} = \epsilon_{22} = \frac{M}{EI}\nu\xi_1, \quad \epsilon_{33} = -\frac{M}{EI}\xi_1. \tag{17}$$

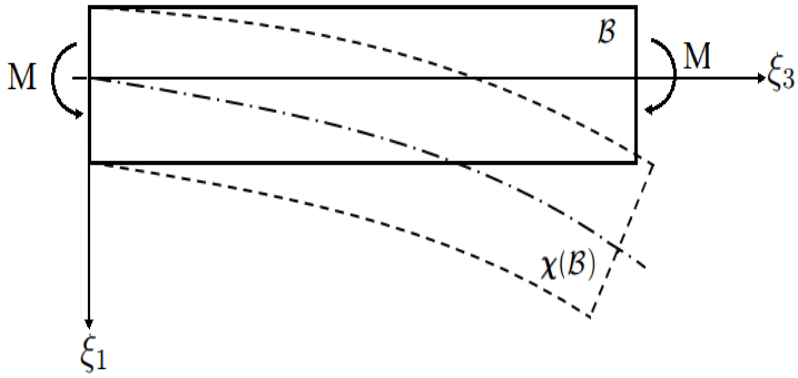

Figure 1: Beam bent by terminal couples.

To use the correspondence relation (12), we first consider that $\mathbf{x}_0$ in (12) is at the origin of the coordinate system and recall from Section 2.1 that N_δ is a sphere with center at this point. We then use (9) together with $\widehat{\mathbf{E}}[\underline{\mathbf{h}}] \equiv \mathbf{E}(\xi_1, \xi_2, \xi_3)$ and the strain components given by (17) to obtain $(M^2/2EI^2)\xi_1^2$ in the integrand on the right hand side of (12). Using the coordinate transformation

$$\xi_1 = \rho\ cos\theta\ sin\phi, \quad \xi_2 = \rho\ sin\theta\ sin\phi, \quad \xi_3 = \rho\ cos\phi, \tag{18}$$

and taking the limits of integration

$$\rho \in (0, \delta), \quad \phi \in (0, \pi), \quad \theta \in (0, 2\pi), \tag{19}$$

we obtain

$$\overline{W}^L_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \frac{M^2 m_6}{6EI^2 m_4}, \tag{20}$$

where m_4 and m_6 are given by the general expression

$$m_n := 4\pi \int_0^\delta \widetilde{\omega}(\rho)\rho^n d\rho, \quad n = 1, 2, \dots . \tag{21}$$

Observe from both (13) and (21) that $m = m_4$.

To determine the expressions of $\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle$ and $\underline{\varphi}\langle\boldsymbol{\xi}\rangle$ in (2) for this experiment, first, we use the definition in (1.b) together with

$$\underline{\mathbf{u}}\langle\boldsymbol{\xi}\rangle := (\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x}_0))\ |_{\mathbf{x}=\mathbf{x}_0+\boldsymbol{\xi}} \tag{22}$$

and recall from above that $\mathbf{x}_0$ is at the origin to obtain $\underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle = \mathbf{u}(\rho, \phi, \theta)/\rho$, where $\mathbf{u}(\rho, \phi, \theta)$ can be obtained from (16) by using the coordinate transformation in (18) together with the change of basis

$$\begin{aligned} &\mathbf{e}_1 = cos\theta\, sin\phi\, \mathbf{e}_\rho + cos\theta\, cos\phi\, \mathbf{e}_\phi - sin\theta\, \mathbf{e}_\theta, \\ &\mathbf{e}_2 = sin\theta\, sin\phi\, \mathbf{e}_\rho + sin\theta\, cos\phi\, \mathbf{e}_\phi + cos\theta\, \mathbf{e}_\theta, \quad \mathbf{e}_3 = cos\phi\, \mathbf{e}_\rho + sin\phi\, \mathbf{e}_\phi. \end{aligned} \tag{23}$$

Using the additive decomposition given by (2), we then get the radial and non-radial components of $\underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle$, which are given by, respectively,

$$\underline{\varphi}\langle\boldsymbol{\xi}\rangle = \frac{M\rho}{2EI} cos\theta \; sin\phi(\nu^2\phi - cos^2\phi), \tag{24}$$

$$\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle = \frac{M\rho}{4EI}\{-cos\phi \; cos\theta[-3-\nu+(1+\nu)cos(2\phi)]\mathbf{e}_\phi + 2sin\theta(\nu^2\phi - cos^2\phi)\mathbf{e}_\theta\}.$$

Next, we substitute the multiplicative decomposition of $\omega(|\boldsymbol{\xi}|, |\boldsymbol{\eta}|)$, given by (11), (24), and the limits of integration in (19) into the expressions (5), (6), and (7) to get

$$\begin{aligned}\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}] =& \hat{\alpha}_{11}\,\frac{mm_6}{840}\frac{M^2}{(EI)^2}(24\nu^2 - 8\nu + 3) + \alpha_{33}\,\frac{mm_6}{6720}\frac{M^2}{(EI)^2}(64\nu^2 + 16\nu + 92) \\ &+ \hat{\alpha}_{13}\,\frac{\pi {m_5}^2}{3360}\frac{M^2}{(EI)^2}(-11\nu^2 + 20\nu - 4),\end{aligned} \tag{25}$$

where both $m \equiv m_4$ and m_6 are given by (21). The terms in (25) that multiply α_{33} and $\hat{\alpha}_{13}$ were calculated by numerical integration.

Substituting the expresion (25) together with (20) into the correspondence relation (12), we can solve the resulting equation for $\hat{\alpha}_{13}$, yielding

$$\hat{\alpha}_{13} = \frac{140}{\pi}\frac{m_6\;\mu}{m\;m_5^2}\frac{8\nu^2 - 8\nu - 1}{11\nu^2 - 20\nu + 4}, \tag{26}$$

where we have used the expressions of $\hat{\alpha}_{11}$, α_{12}, and α_{33} given by (14) and (15).

3 VALIDATION OF EXPRESSIONS FOR MATERIAL CONSTANTS

So far, we have obtained the expressions in both (14) and (15) by considering homogeneous deformations in [1] and the uniform torsion of a circular shaft in [2]. We have also obtained (26) by considering the beam bent by terminal couples in [3]. In that work, $\mathbf{x}_0$ is at the origin of the left end of the beam. To verify the validity of these expressions, we now consider that $\mathbf{x}_0$ is an arbitrary point of the beam, calculate both the peridynamic free energy function using the expressions of the peridynamic constants, given by (14), (15), and (26), and the classical strain energy density at $\mathbf{x}_0$ and then verify that the correspondence relation (12) is satisfied.

To obtain the infinitesimal strain tensor $\mathbf{E}$ evaluated at $\mathbf{x}_0 = (x_0,\, y_0,\, z_0)$, we substitute

$$\xi_1 = \widehat{\xi}_1 + x_0, \quad \xi_2 = \widehat{\xi}_2 + y_0, \quad \xi_3 = \widehat{\xi}_3 + z_0 \tag{27}$$

into the expressions in (17), where $(\widehat{\xi}_1, \widehat{\xi}_2, \widehat{\xi}_3)$ are the components of the relative position vector $\boldsymbol{\xi}$. Using the coordinate transformations in (18) for $(\widehat{\xi}_1,\, \widehat{\xi}_2,\, \widehat{\xi}_3)$ and taking the limits of integration in (19), we obtain

$$\overline{W}^L_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \frac{M^2}{EI^2}\left(\frac{{x_0}^2}{2} + \frac{m_6}{6m}\right), \tag{28}$$

where both $m = m_4$ and m_6 are given by (21). Observe from (28) that only the first term within the parentheses depends upon the position $\mathbf{x}_0$ and that the remaining term yields (20).

Next, we use (1.b) together with (22), where $\mathbf{u}(\mathbf{x})$ is given by (16), to obtain

$$\begin{aligned}\underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle =& \frac{M}{2EI\,|\boldsymbol{\xi}|}\Big(\Big\{\widehat{\xi}_3\,(2z_0+\widehat{\xi}_3)+\nu\left[\widehat{\xi}_1^2+2z_0\,\widehat{\xi}_1-\widehat{\xi}_2\,(2y_0+\widehat{\xi}_2)\right]\Big\}\,\mathbf{e}_1\\ &+2\nu\left[y_0\,\widehat{\xi}_1+(x_0+\widehat{\xi}_1)\widehat{\xi}_2\right]\mathbf{e}_2-2\left[z_0\widehat{\xi}_1+(x_0+\widehat{\xi}_1)\widehat{\xi}_3\right]\mathbf{e}_3\Big).\end{aligned} \tag{29}$$

Similarly as before, we use (2) to decompose $\underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle$, given by (29), into radial and non-radial components, given by $\underline{\varphi}\langle\boldsymbol{\xi}\rangle = \underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle\cdot\underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle$ and $\underline{\mathbf{h}}_d\langle\boldsymbol{\xi}\rangle = \underline{\mathbf{h}}\langle\boldsymbol{\xi}\rangle - \underline{\varphi}\langle\boldsymbol{\xi}\rangle\underline{\mathbf{e}}\langle\boldsymbol{\xi}\rangle$, respectively. Using the coordinate transformation in (18) to write these expressions in spherical coordinates, substituting the resulting expressions in the integrands of (5) thru (7), and using the limits of integration in (19), we obtain

$$\begin{aligned}\widehat{W}_{\mathbf{x}_0}[\underline{\varphi}\,\underline{\mathbf{e}}] =& \frac{\hat{\alpha}_{11}}{2}\left(\frac{M}{EI}\right)^2\left[m^2\frac{1}{15}(3-4\nu+8\nu^2){x_0}^2+m_6\,m\,\frac{3-8\nu+24\nu^2}{420}\right]\\ &+\alpha_{12}\left(\frac{M}{EI}\right)^2\frac{m^2}{9}(1-2\nu)^2{x_0}^2.\end{aligned} \tag{30}$$

Observe from (30) that the second term inside the square brackets yields the expression which multiplies $\hat{\alpha}_{11}$ in (25) and that the remaining terms are position dependent and proportional to ${x_0}^2$.

We use numerical integration to calculate the integrals that multiply $\hat{\alpha}_{13}$ in (5) and α_{33} in (7). We divide the integrands into ten parts that correspond to multiplications by the ten monomials 1, x_0, y_0, z_0, ${x_0}^2$, ${y_0}^2$, ${z_0}^2$, $x_0\,y_0$, $x_0\,z_0$, and $y_0\,z_0$. Then, we divide again each one of these parts into terms that multiply 1, ν, and ν^2, and, finally, integrate numerically each one of the resulting parts.

Concerning the integrals that multiply $\hat{\alpha}_{13}$ above, the ones that multiply terms in the set $\{x_0,\, y_0,\, z_0,\, {x_0}^2,\, {y_0}^2,\, {z_0}^2,\, x_0\,y_0,\, x_0\,z_0,\, y_0\,z_0\}$ are nearly zero when compared to the remaining three integrals that multiply 1. These integrals yield

$$\left[\frac{\pi {m_5}^2}{3360}\frac{M^2}{(EI)^2}(-11\nu^2+20\nu-4)\right], \tag{31}$$

which is the same expression that multiplies $\hat{\alpha}_{13}$ in (25).

Concerning the integrals that multiply α_{33} above, in addition to the position independent terms, also the terms that multiply ${x_0}^2$ do not vanish. A procedure similar to the one presented above for the calculation of the terms that multiply $\hat{\alpha}_{13}$ yields

$$\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}_d] = \alpha_{33}\frac{M^2}{(EI)^2}\left[m^2\,(1+\nu)^2\frac{{x_0}^2}{45}+\frac{m\,m_6}{64}\frac{(64\nu^2+16\nu+92)}{105}\right]. \tag{32}$$

Replacing (30), (31), and (32) into (5) and using the expressions in (14), (15), and (26) for the peridynamic constants, we finally obtain

$$\widehat{W}_{\mathbf{x}_0}[\underline{\mathbf{h}}] = \frac{M^2}{EI^2}\left(\frac{{x_0}^2}{2}+\frac{m_6}{6m}\right), \tag{33}$$

which is equal to $\widehat{W}^{L}_{\mathbf{x}_0}[\underline{\mathbf{h}}]$ in (28), satisfying, therefore, the correspondence relation (12).

4 CONCLUDING REMARKS

The four peridynamic constants that appear in the free energy function defined by (5) together with (6) and (7) were previously determined using homogeneous deformations, uniform torsion of a circular shaft and bending of a beam by terminal couples. In this last experiment results were obtained at a single point of the beam. In this work, we have verified the validity of the closed form expressions for the peridynamic constants by considering an arbitrary point of the beam. At the conference we will show results that further validate the closed form expressions. These results were obtained from displacement fields from classical linear elasticity for both beam bent by terminal load and circular shaft subjected to anti-plane shear.

In summary, all the results indicate that the correspondence relation (12) is nearly satisfied for the above experiments. We then see that the values of the peridynamic constants are not dependent on the type of experiment. This work is, therefore, an important contribution to the development of a three-dimensional state-based peridynamic theory.

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