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THE BREZIS-EKELAND-NAYROLES VARIATIONAL PRINCIPALE FOR ELASTOPLASTIC PROBLEMS

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Abstract. Using the concept of symplectic subdifferential, a modification of the Hamiltonian formalism which can be used for dissipative systems is proposed. The formalism is specialized to the standard plasticity in small strains and statics. It is applied to solve the classical problem of a thick tube in plane strain subjected to an internal pressure. The continuum is discretized with mixed finite elements.

1 INTRODUCTION

Realistic dynamical systems considered by engineers and physicists are subjected to energy loss. It may stem from external actions, the conservative case. The behaviour of such systems can be represented by Hamilton's least action principle. If the cause is internal, resulting from a broad spectrum of phenomena such as collisions, surface friction, viscosity, plasticity, fracture, damage and so on, it's called dissipative. Hamilton's variational principle doesn't work for such systems, so another principle is proposed.

Classical dynamics are generally addressed through the world of smooth functions while the mechanics of dissipative systems deals with the one of non smooth functions. Unfortunately, both worlds widely ignore each other. Aim of this article is laying strong foundations to link both worlds and their corresponding methods.

2 NON DISSIPATIVE SYSTEMS

The variables of a dynamical systems are $z = (x, y) \in X \times Y$ where the degrees of freedom x describe the body motion and y are the corresponding momenta. X and Y are topological, locally convex, real vector spaces. There is a dual pairing $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$ which makes continuous the linear forms $x \mapsto \langle x, y \rangle$ and $y \mapsto \langle x, y \rangle$.

The space $X \times Y$ has a natural symplectic form $\omega : (X \times Y)^2 \to \mathbb{R}$ defined by :

$$\omega(z, z') = \langle x, y' \rangle - \langle x', y \rangle$$

For any smooth hamiltonian function $(x, t) \mapsto H(x, t)$, the symplectic gradient (or Hamiltonian vector field) is defined by $\dot{z} = XH \Leftrightarrow \forall \delta z, \, \omega(\dot{z}, \delta z) = \delta H$. In the particular case

$X = Y$, the dual pairing is a scalar product and the space $X \times Y$ is dual with itself, with the duality product :

$$\langle\langle (x, y), (x', y') \rangle\rangle \ = \ \langle x, x' \rangle + \langle y, y' \rangle$$

Introducing the linear map $J(x, y) = (-y, x)$ and putting $\omega(z, z') = \langle\langle J(z), z' \rangle\rangle$ which allows to recover the canonical equations governing the motion :

$$\dot{x} = grad_y H, \qquad \dot{y} = -grad_x H \tag{1}$$

3 DISSIPATIVE SYSTEMS

For such systems, the cornerstone hypothesis is to decompose the velocity in the phase space into reversible and irreversible parts :

$$\dot{z} \ = \ \dot{z}_R + \dot{z}_I, \quad \dot{z}_R = X\, H, \quad \dot{z}_I \ = \ \dot{z} \ - \ X\, H$$

For a non dissipative system, the irreversible part vanishes and the motion is governed by the canonical equations. A crucial turning-point is the tools of the differential geometry to the ones of the non smooth mechanics. Starting with a dissipation potential ϕ, it is not differentiable everywhere but convex and lower semicontinuous. Introducing a new subdifferential, called symplectic [1]. Mere sleight of hand, all what have to do is to replace the dual pairing by the symplectic form in the classical definition :

$$\dot{z}_I \in \partial^\omega \phi(\dot{z}) \qquad \Leftrightarrow \qquad \forall \dot{z}', \quad \phi(\dot{z} + \dot{z}') - \phi(\dot{z}) \geq \omega(\dot{z}_I, \dot{z}') \tag{2}$$

From a mechanical viewpoint, it is the constitutive law of the material. Likewise, defining a symplectic conjugate function, by the same sleight of hand in the definition of the Legendre-Fenchel transform :

$$\phi^{*\omega}(\dot{z}_I) = \sup_{\dot{z}} \{\omega(\dot{z}_I, \dot{z}) - \phi(\dot{z})\}$$

satisfying a symplectic Fenchel inequality :

$$\phi(\dot{z}) + \phi^{*\omega}(\dot{z}_I) - \omega(\dot{z}_I, \dot{z}) \geq 0 \tag{3}$$

The equality is reached in the previous relation if and only if the constitutive law (2) is satisfied.

Remarks. Always in the case $X = Y$, taking into account the antisymmetry of ω :

$$\langle\langle D_z H, \dot{z} \rangle\rangle = \langle\langle J\,(X\, H), \dot{z} \rangle\rangle = \omega(X\, H, \dot{z}) = \omega(\dot{z}, \dot{z} - X\, H) = \omega(\dot{z}, \dot{z}_I)$$

If one supposes that for all couples $(\dot{z}, \dot{z}_I)$:

$$\phi(\dot{z}) + \phi^{*\omega}(\dot{z}_I) \geq 0$$

the system dissipates for the couples satisfying the constitutive law :

$$\langle\langle D_z H, \dot{z} \rangle\rangle = -\omega(\dot{z}_I, \dot{z}) = -(\phi(z, \dot{z}) + \phi^{*\omega}(z, \dot{z}_I)) \leq 0$$

4 THE SYMPLECTIC BREZIS-EKELAND-NAYROLES PRINCIPLE

The variational formulation can be obtained by integrating the left hand member of (3) on the system evolution. On this ground, a symplectic version of the Brezis-Ekeland-Nayroles (BEN) variational principle [2] is proposed :

The natural evolution curve $z : [t_0, t_1] \to X \times Y$ *minimizes the functional :*

$$\Pi(z) := \int_{t_0}^{t_1} [\phi(\dot{z}) + \phi^{*\omega}(\dot{z} - XH) - \omega(\dot{z} - XH, \dot{z})] \; \mathrm{dt}$$

among all the curves verifying the initial conditions $z(t_0) = z_0$ *and, remarkably, the minimum is zero.* Observing that $\omega(\dot{z}, \dot{z})$ vanishes and integrating by part, the variant is given (which is not compulsory) :

$$\Pi(z) = \int_{t_0}^{t_1} [\phi(\dot{z}) + \phi^{*\omega}(\dot{z} - XH) - \frac{\partial H}{\partial t}(t, z)] \; \mathrm{dt} + H(t_1, z(t_1)) - H(t_0, z_0)$$

5 APPLICATION TO THE STANDARD PLASTICITY AND VISCOPLASTICITY

To illustrate the general formalism and to show how it allows to develop powerful variational principles for dissipative systems within the frame of continuum mechanics, the standard plasticity and viscoplasticity in small deformations based on the additive decomposition of strains into reversible and irreversible strains ($\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_R + \boldsymbol{\varepsilon}_I$ where $\boldsymbol{\varepsilon}_I$ is the plastic strain) are studied. Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set, with piecewise smooth boundary $\partial\Omega$. As usual, it is divided into two disjoint parts, $\partial\Omega_0$ (called support) where the displacements are imposed and $\partial\Omega_1$ where the surface forces are imposed. The elements of the space X are fields $x = (\boldsymbol{u}, \boldsymbol{\varepsilon}_I) \in U \times E$ where $\boldsymbol{u}$ is a displacement field on the body Ω with trace $\bar{\boldsymbol{u}}$ on $\partial\Omega$. The elements of the corresponding dual space Y are of the form $y = (\boldsymbol{p}, \boldsymbol{\pi})$. Unlike $\boldsymbol{p}$ which is clearly the linear momentum, the physical meaning of $\boldsymbol{\pi}$ is not known at this stage.

The duality between the spaces X and Y has the form :

$$\langle x, y \rangle \; = \; \int_\Omega (\langle \boldsymbol{u}, \boldsymbol{p} \rangle + \langle \boldsymbol{\varepsilon}_I, \boldsymbol{\pi} \rangle)$$

where the duality products which appear in the integral are finite dimensional duality products on the image of the fields $\boldsymbol{u}, \boldsymbol{p}$ (for our example this means a scalar product on $\mathbb{R}^3$) and on the image of the fields $\boldsymbol{\varepsilon}, \boldsymbol{\pi}$ (in this case this is a scalar product on the space of 3 by 3 symmetric matrices). All these standard dualities are denoted by the same $\langle \cdot, \cdot \rangle$ symbols.

The total Hamiltonian of the structure is taken of the integral form :

$$H(t, z) = \int_\Omega \left\{ \frac{1}{2\rho} \parallel \boldsymbol{p} \parallel^2 + w(\nabla \boldsymbol{u} - \boldsymbol{\varepsilon}_I) - \boldsymbol{f}(t) \cdot \boldsymbol{u} \right\} - \int_{\partial\Omega_1} \bar{\boldsymbol{f}}(t) \cdot \boldsymbol{u}$$

The first term is the kinetic energy, w is the elastic strain energy, $\boldsymbol{f}$ is the volume force and $\bar{\boldsymbol{f}}$ is the surface force on the part $\partial\Omega_1$ of the boundary, the displacement field being equal to an imposed value $\bar{\boldsymbol{u}}$ on the remaining part $\partial\Omega_0$.

According to (1), its symplectic gradient is $XH = ((D_{\boldsymbol{p}}H, D_{\boldsymbol{\pi}}H), (-D_{\boldsymbol{u}}H, -D_{\boldsymbol{\varepsilon}_I}H))$ where, introducing as usual the stress field $\boldsymbol{\sigma} = Dw(\nabla \boldsymbol{u} - \boldsymbol{\varepsilon}_I)$ $D_{\boldsymbol{u}}H$ which is the gradient in the variational sense (from (1) and the integral form of the duality product) :

$$D_{\boldsymbol{u}}H = \frac{\partial H}{\partial \boldsymbol{u}} - \nabla \cdot \left(\frac{\partial H}{\partial \nabla \boldsymbol{u}} \right) = -\boldsymbol{f} - \nabla \cdot \boldsymbol{\sigma} \quad D_{\bar{\boldsymbol{u}}}H = \boldsymbol{\sigma} \cdot \boldsymbol{n} - \bar{\boldsymbol{f}}$$

Thus one has :

$$\dot{z}_I = \dot{z} - XH = \left(\left(\dot{\boldsymbol{u}} - \frac{\boldsymbol{p}}{\rho}, \dot{\boldsymbol{\varepsilon}}_I \right), (\dot{\boldsymbol{p}} - \boldsymbol{f} - \nabla \cdot \boldsymbol{\sigma}, \dot{\boldsymbol{\pi}} - \boldsymbol{\sigma}) \right)$$

A dissipation potential which has an integral form is given as $\Phi(z) = \int_\Omega \phi(\boldsymbol{p}, \boldsymbol{\pi})$ and one assumes the symplectic Fenchel transform of Φ expresses as the integral of the symplectic Fenchel transform of the dissipation potential density ϕ.

The symplectic Fenchel transform of the function ϕ reads :

$$\phi^{*\omega}(\dot{z}_I) = \sup \left\{ \langle \dot{\boldsymbol{u}}_I, \dot{\boldsymbol{p}}' \rangle + \langle \dot{\boldsymbol{\varepsilon}}_I, \dot{\boldsymbol{\pi}}' \rangle - \langle \dot{\boldsymbol{u}}', \dot{\boldsymbol{p}}_I \rangle - \langle \dot{\boldsymbol{\varepsilon}}'_I, \dot{\boldsymbol{\pi}}_I \rangle - \phi(\dot{z}') \, : \, \dot{z}' \in X \times Y \right\}$$

To recover the standard plasticity, supposing that ϕ is depending explicitly only on $\dot{\boldsymbol{\pi}}$:

$$\phi(\dot{z}) = \varphi(\dot{\boldsymbol{\pi}}) \tag{4}$$

Denoting by χ_K the indicator function of a set K (equal to 0 on K and to $+\infty$ otherwise) : $\phi^{*\omega}(\dot{z}_I) = \chi_{\{\mathbf{0}\}}(\dot{\boldsymbol{u}}_I) + \chi_{\{\mathbf{0}\}}(\dot{\boldsymbol{p}}_I) + \chi_{\{\mathbf{0}\}}(\dot{\boldsymbol{\pi}}_I) + \varphi^*(\dot{\boldsymbol{\varepsilon}}_I)$ where φ^* is the usual Fenchel transform. In other words, the quantity $\phi^{*\omega}(\dot{z}_I)$ is finite and equal to $\phi^{*\omega}(\dot{z}_I) = \varphi^*(\dot{\boldsymbol{\varepsilon}}_I)$ if and only if all of the following are true :

(a) $\boldsymbol{p}$ equals the linear momentum

$$\boldsymbol{p} = \rho \dot{\boldsymbol{u}} \tag{5}$$

(b) The balance of linear momentum is satisfied

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{f} = \dot{\boldsymbol{p}} = \rho \ddot{\boldsymbol{u}} \quad \text{on} \quad \Omega, \qquad \boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{f}} \quad \text{on} \quad \partial\Omega_1 \tag{6}$$

(c) An equality which reveals the meaning of the variable $\boldsymbol{\pi}$:

$$\dot{\boldsymbol{\pi}} = \boldsymbol{\sigma} \tag{7}$$

Eliminating $\dot{\boldsymbol{\pi}}$ by (7), the symplectic BEN principle applied to standard plasticity states that the evolution curve minimizes :

$$\Pi(z) = \int_{t_0}^{t_1} \left\{ \varphi(\boldsymbol{\sigma}) + \varphi^*(\dot{\boldsymbol{\varepsilon}}_I) - \frac{\partial H}{\partial t}(t, z) \right\} \, \mathrm{d}t + H(t_1, z(t_1)) - H(t_0, z_0) \tag{8}$$

among all curves $z : [t_0, t_1] \to X \times Y$ such that $z(0) = (x_0, y_0)$, the kinematical conditions on $\partial\Omega_0$, (5), (6) are satisfied. For instance, in plasticity, the potential φ is the indicator function of the plastic domain.

Remark. The assumption that $\boldsymbol{u}, \boldsymbol{\varepsilon}_I$ and $\boldsymbol{p}$ are ignorable in (4) comes down to introduce into the dynamical formalism a "statical" constitutive law : $\dot{\boldsymbol{\varepsilon}}_I \in \partial\varphi(\dot{\boldsymbol{\pi}}) = \partial\varphi(\boldsymbol{\sigma})$. Conversely, the symplectic framework suggests to imagine fully *"dynamical' constitutive laws* of the more general form : $(\dot{\boldsymbol{u}}, \dot{\boldsymbol{\varepsilon}}_I) \in \partial^\omega \phi\, (\dot{\boldsymbol{p}}, \dot{\boldsymbol{\pi}})$.

The symplectic BEN principle and the original BEN principle. Examining the important case where the kinetic energy and inertia forces can be neglected (quasi-static behaviour) gives :

$$\dot{\boldsymbol{p}} = \mathbf{0}, \qquad H(t,z) = \int_\Omega \{w(\nabla \boldsymbol{u} - \boldsymbol{\varepsilon}_I) - \boldsymbol{f}(t)\cdot\boldsymbol{u}\} - \int_{\partial\Omega_1} \bar{\boldsymbol{f}}(t)\cdot\boldsymbol{u}$$

and the elasticity is linear :

$$\dot{\boldsymbol{\varepsilon}}_I = \nabla \dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}} \tag{9}$$

denoting $\boldsymbol{S} = (Dw)^{-1}$ the compliance operator. Eliminating $\boldsymbol{\pi}$ and $\boldsymbol{p}$ thanks to (5) and (7), the symplectic BEN principle (8) is transformed and claims that the evolution curve minimizes :

$$\Pi(\boldsymbol{\sigma}, \dot{\boldsymbol{u}}) = \int_{t_0}^{t_1} \left\{ \varphi(\boldsymbol{\sigma}) + \varphi^*(\nabla\dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}}) - \frac{\partial H}{\partial t}(t,z) \right\} \,\mathrm{dt} + H(t_1, z(t_1)) - H(t_0, z_0) \tag{10}$$

among all curves $\boldsymbol{u} : [t_0, t_1] \to U$ satisfying the kinematical conditions on $\partial\Omega_0$ and all curves $\boldsymbol{\sigma} : [t_0, t_1] \to E$ such that $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ and

$$\nabla\cdot\boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0} \quad \text{on} \quad \Omega, \qquad \boldsymbol{\sigma}\cdot\boldsymbol{n} = \bar{\boldsymbol{f}} \quad \text{on} \quad \partial\Omega_1 \tag{11}$$

are satisfied. This expression can be transformed as follows for sake of easiness :

$$\langle \boldsymbol{l}(t), u\rangle = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{u} + \int_{\partial\Omega_1} \bar{\boldsymbol{f}}(t)\cdot\boldsymbol{u}$$

Then $\frac{\partial H}{\partial t}(t,z) = -\langle \dot{\boldsymbol{l}}(t), \boldsymbol{u}\rangle$. In the other hand : $\frac{d}{dt}\left[H(t,z(t))\right] = \langle \boldsymbol{\sigma}, \nabla\dot{\boldsymbol{u}} - \dot{\boldsymbol{\varepsilon}}_I\rangle - \langle \boldsymbol{l}(t), \dot{\boldsymbol{u}}\rangle - \langle \dot{\boldsymbol{l}}(t), u\rangle$. For the minimizer, the kinematical conditions on $\partial\Omega_0$ and the equilibrium equations (11) are satisfied and using Green's formula :

$$\langle \boldsymbol{\sigma}, \nabla\dot{\boldsymbol{u}}\rangle = \langle \boldsymbol{l}(t), \dot{\boldsymbol{u}}\rangle \tag{12}$$

that leads to $\frac{d}{dt}\left[H(t,z(t))\right] - \frac{\partial H}{\partial t}(t,z) = -\langle \boldsymbol{\sigma}, \dot{\boldsymbol{\varepsilon}}_I\rangle$.

Time-integrating, replacing in (10) and taking into account (9) leads to the original BEN principle [3, 4]. The evolution curve minimizes :

$$\Pi(\boldsymbol{\sigma}, , \dot{\boldsymbol{u}}) = \int_{t_0}^{t_1} \{\varphi(\boldsymbol{\sigma}) + \varphi^*(\nabla\dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}}) - \langle \boldsymbol{\sigma}, \nabla\dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}}\rangle\} \,\mathrm{dt} \tag{13}$$

among all curves $\boldsymbol{u} : [t_0, t_1] \to U$ satisfying the kinematical conditions on $\partial\Omega_0$ and all curves $\boldsymbol{\sigma} : [t_0, t_1] \to E$ and the equilibrium equations (11) are satisfied.

6 THE THICK TUBE PROBLEM

A thick tube which has internal radius a and external one b in plane strain and is subjected to an internal pressure $p > 0$ monotonic increasing from zero is studied. The material is elastic perfectly plastic and isotropic with Tresca model and yield stress σ_Y. The initial stresses and displacements are null. To ensure in statical equilibrium, the pressure must not overcome the limit value. And the dissipation potential is: $\varphi(\boldsymbol{\sigma}) = \int_\Omega \chi_K(\boldsymbol{\sigma})$.

The elastic domain is shown below with $\boldsymbol{s}$ the stress deviator tensor :

$$K = \{\boldsymbol{\sigma} \quad \text{such that} \quad \sigma_{\theta\theta} - \sigma_{rr} - \sigma_Y \leq 0\}$$

The inelastic strain rate $\dot{\boldsymbol{\varepsilon}}_I$ is plastic and denoted $\dot{\boldsymbol{\varepsilon}}^p$. The Fenchel conjugate is obtained combining this rule with the expression of the dissipation power by unit volume and the yield condition :

$$D = \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}}^p = \sigma_Y \dot{\varepsilon}^p_{\theta\theta}$$

As $\dot{\varepsilon}^p_{\theta\theta}$ must be non negative, the Fenchel conjugate function is :

$$\varphi^*(\dot{\boldsymbol{\varepsilon}}^p) = \int_\Omega \left\{\sigma_Y \dot{\varepsilon}^p_{\theta\theta} + \chi_{\mathbb{R}_+}(\dot{\varepsilon}^p_{\theta\theta})\right\}$$

In plane strain and axisymmetry, the displacement is radial. Taking into account the previous assumptions, the functional (13) becomes :

$$\bar{\Pi}(\boldsymbol{\sigma}, \boldsymbol{u}) = \int_{t_0}^{t_1} \left\{ (\int_\Omega \chi_K(\boldsymbol{\sigma}) + \sigma_Y \dot{\varepsilon}^p_{\theta\theta} + \chi_{\mathbb{R}_+}(\dot{\varepsilon}^p_{\theta\theta}) - \langle \boldsymbol{\sigma}, \nabla \dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}} \rangle \right\} \text{ dt}$$

In this problem, there is no supports ($\partial\Omega_0 = \emptyset$).

As the minimum is certainly finite, this amounts to minimizing :

$$\bar{\Pi}(\boldsymbol{\sigma}, \boldsymbol{u}) = \int_{t_0}^{t_1} \left\{ (\int_\Omega \sigma_Y \dot{\varepsilon}^p_{\theta\theta}) - \langle \boldsymbol{\sigma}, \nabla \dot{\boldsymbol{u}} - \boldsymbol{S}\dot{\boldsymbol{\sigma}} \rangle \right\} \text{ dt} \tag{14}$$

among all the curves among all curves $(\boldsymbol{\sigma}, \boldsymbol{u}) : [t_0, t_1] \to U \times E$ such that $\boldsymbol{\sigma}(0) = \mathbf{0}$, $\boldsymbol{u}(0) = \mathbf{0}$, satisfying the Tresca yield condition. The normality rule and the equilibrium equations :

$$\frac{d}{dr}(r\,\sigma_{rr}) = \sigma_{\theta\theta} \quad \text{for} \quad a < r < b, \qquad \sigma_{rr}(a,t) = -p(t), \qquad \sigma_{rr}(b,t) = 0 \tag{15}$$

7 MIXED FINITE ELEMENT OF THICK TUBE

The continuum is discretized with mixed finite elements. The continuum mechanics requires only the continuity of the radial stress across a section $r = C^{te}$ but the hoop stress is also continuous for the exact solution, an axisymmetric element occupying a volume $\alpha < r < \beta$ with four stress connectors is proposed :

$$g_1 = \sigma_{rr}\mid_{r=\alpha}, \qquad g_2 = \sigma_{\theta\theta}\mid_{r=\alpha}, \qquad g_3 = \sigma_{rr}\mid_{r=\beta}, \qquad g_4 = \sigma_{\theta\theta}\mid_{r=\beta} \tag{16}$$

The choice of a polynomial statically admissible stress field is guided by the aim to avoid the global (or structural) equilibrium equations in the constrained minimization problem. Only remains the local yield condition.

For the stress field being defined by the four connectors : $\sigma_{rr} = h_1 + h_2 r + h_3 r^2 + h_4 r^3$. Using the internal equilibrium equation in (15), the hoop stress is : $\sigma_{\theta\theta} = h_1 + 2\,h_2 r + 3\,h_3 r^2 + 4\,h_4 r^3$. In matrix form, the stress field in terms of stress parameters reads :

$$\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \end{bmatrix} = \boldsymbol{\sigma}_e(r) = \boldsymbol{R}_e(r)\,\boldsymbol{h}_e = \begin{bmatrix} 1 & r & r^2 & r^3 \\ 1 & 2\,r & 3\,r^2 & 4\,r^3 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix}$$

The stress connectors (16) are linearly depending on the stress parameters: $\boldsymbol{g}_e = \boldsymbol{C}_e \boldsymbol{h}_e$. With the connection matrix :

$$\boldsymbol{C}_e = \begin{bmatrix} 1 & \alpha & \alpha^2 & \alpha^3 \\ 1 & 2\,\alpha & 3\,\alpha^2 & 4\,\alpha^3 \\ 1 & \beta & \beta^2 & \beta^3 \\ 1 & 2\,\beta & 3\,\beta^2 & 4\,\beta^3 \end{bmatrix}$$

Eliminating the stress parameters provides the stress field in terms of stress connectors :

$$\boldsymbol{\sigma}(r) = \boldsymbol{R}_e(r)\boldsymbol{C}_e^{-1}\boldsymbol{g}_e = \boldsymbol{T}_e(r)\boldsymbol{g}_e$$

The displacement field is proposed : $u_r = u_1 + u_2 r + u_3 r^2 + u_4 r^3$ to provide a strain field with the same number of parameters as the one of the stress field.

$$\varepsilon_{rr} = \frac{du_r}{dr} = u_2 + 2\,u_3 r + 3\,u_4 r^2, \qquad \varepsilon_{\theta\theta} = \frac{u_r}{r} = \frac{u_1}{r} + u_2 + u_3 r + u_4 r^2$$

There is two connectors: $q_1 = u_r \mid_{r=\alpha}$ and $q_2 = u_r \mid_{r=\beta}$. Considering two intermediate equidistant nodes of position :

$$\gamma = \frac{2\,\alpha + \beta}{3}, \qquad \delta = \frac{\alpha + 2\,\beta}{3}$$

Introducing two extra degrees of freedom internal to the element (not connected with the other ones) : $q_3 = u_r \mid_{r=\gamma}$ and $q_4 = u_r \mid_{r=\delta}$. By defining a cubic Lagrange interpolation, one has :

$$u_r(r) = \boldsymbol{N}_e(r)\,\boldsymbol{q}_e$$

with :

$$\boldsymbol{N}_e^T(r) = \frac{1}{16} \begin{bmatrix} -(1-\rho)\,(1-9\,\rho^2) \\ -(1+\rho)\,(1-9\,\rho^2) \\ 9\,(1-\,\rho^2)\,(1-3\,\rho) \\ 9\,(1-\,\rho^2)\,(1+3\,\rho) \end{bmatrix}$$

where $\rho = \frac{2\,r-(\beta+\alpha)}{\beta-\alpha}$. The corresponding strain field can be expressed in term of the nodal displacement :

$$\boldsymbol{\varepsilon}(r) = \begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \dfrac{d\boldsymbol{N_e}}{d\rho}\dfrac{d\rho}{dr} \\ \dfrac{\boldsymbol{N_e}}{r} \end{bmatrix} \boldsymbol{q}_e = \boldsymbol{B}_e(r)\,\boldsymbol{q}_e$$

After calculation, one has:

$$\boldsymbol{B}_e(r) = \frac{1}{16} *$$

$$\begin{bmatrix} J\,(1+18\,\rho-27\,\rho^2) & J\,(-1+18\,\rho+27\,\rho^2) & J\,(-27-18\,\rho+81\,\rho^2) & J\,(27-18\,\rho-81\,\rho^2) \\ -\dfrac{1}{r}(1-\rho)\,(1-9\,\rho^2) & -\dfrac{1}{r}(1+\rho)\,(1-9\,\rho^2) & \dfrac{9}{r}\,(1-\ \rho^2)\,(1-3\,\rho) & \dfrac{9}{r}\,(1-\ \rho^2)\,(1+3\,\rho) \end{bmatrix}$$

with $J = \dfrac{d\rho}{dr} = \frac{2}{\beta-\alpha}$.

7.1 Space discretization of the principle

As usual, the integral are approximated by numerical integration ($g = 4$ for example) on every element :

$$\int_{\alpha}^{\beta} \boldsymbol{A}(r)\,2\,\pi\,r\,dr \cong \sum_{g=1}^{4} w_g\,\boldsymbol{A}(r_g)\,2\,\pi\,r_g$$

So the total dissipation power in the element reads $\int_{\alpha}^{\beta} D(r)\,2\,\pi\,r\,dr = \Lambda_g^T\,\dot{\lambda}_g$ with :

$$\boldsymbol{\Lambda}_e = \begin{bmatrix} w_1\,2\,\pi\,r_1 \\ \cdots \\ w_4\,2\,\pi\,r_4 \end{bmatrix}, \qquad \dot{\boldsymbol{\lambda}}_e = \begin{bmatrix} \dot{\lambda}_1 \\ \cdots \\ \dot{\lambda}_4 \end{bmatrix}$$

Performing the assembling thanks to the localization matrices $\boldsymbol{M}_e, \boldsymbol{L}_e, \boldsymbol{P}_e$ such that :

$$\boldsymbol{g}_e = \boldsymbol{M}_e\boldsymbol{g} \qquad \boldsymbol{q}_e = \boldsymbol{L}_e\boldsymbol{q}, \qquad \dot{\boldsymbol{\lambda}}_e = \boldsymbol{P}_e\dot{\boldsymbol{\lambda}}$$

the discretized form of the functional (14) is :

$$\bar{\Pi}(\boldsymbol{g},\boldsymbol{q},\dot{\boldsymbol{\lambda}}) = \int_{t_0}^{t_1} (\boldsymbol{\Lambda}^T\dot{\boldsymbol{\lambda}}(t) - \dot{\boldsymbol{q}}^T(t)\boldsymbol{G}\,\boldsymbol{g}(t) + \dot{\boldsymbol{g}}^T(t)\boldsymbol{F}\,\boldsymbol{g}(t))\ \mathrm{dt} \tag{17}$$

with :

$$\boldsymbol{\Lambda} = \sum_{e=1}^{n} \boldsymbol{P}_e^T\boldsymbol{\Lambda}_e,$$

$$\boldsymbol{G} = \sum_{e=1}^{n} \int_{\alpha}^{\beta} \boldsymbol{L}_e^T\boldsymbol{B}_e^T(r)\,\boldsymbol{T}_e(r)\,\boldsymbol{M}_e\,2\,\pi\,r\,dr, \qquad \boldsymbol{F} = \sum_{e=1}^{n} \int_{\alpha}^{\beta} \boldsymbol{M}_e^T\boldsymbol{T}_e^T(r)\,\boldsymbol{S}\,\boldsymbol{T}_e(r)\,\boldsymbol{M}_e\,2\,\pi\,r\,dr$$

The BEN claims that it needs to find the minimum of (17) with respect to the path $t \mapsto (\boldsymbol{g}(t), \boldsymbol{q}(t), \dot{\boldsymbol{\lambda}}(t))$ under the constrains of :

- Equilibrium (on the boundary, the internal equilibrium being satisfies *a priori*) :

$$g_1(t) = -p(t), \quad g_{2\,(n+1)}(t) = 0$$

- Plasticity (at every integration point g of every element e) :

$$\boldsymbol{N}_{Y,e}^T(r_g)\boldsymbol{g} - \sigma_Y \leq 0, \quad \dot{\lambda}_g \geq 0, \quad \boldsymbol{N}_Y \dot{\lambda}_g = \boldsymbol{B}_e(r_g)\,\dot{\boldsymbol{q}}_e - \boldsymbol{S}\,\boldsymbol{T}_e(r_g)\,\dot{\boldsymbol{g}}_e$$

- Initial conditions :

$$\boldsymbol{g}(t_0) = \boldsymbol{0}, \qquad \boldsymbol{q}(t_0) = \boldsymbol{0}, \qquad \dot{\boldsymbol{\lambda}}(t_0) = \boldsymbol{0}$$

With $\boldsymbol{N}_{Y,e}(r) = \boldsymbol{M}_e^T \boldsymbol{T}_e^T(r)\boldsymbol{N}_Y$, $\boldsymbol{N}_Y^T = [-1 \quad 1]$.

7.2 Time discretization of the functional

For the time discretization of any physical quantity a, putting $a_j = a(t_j)$, $\Delta a_j = a_j - a_{j-1}$. On each step, one approximates the time rates by $\dot{a} = \frac{\Delta a_j}{\Delta t_j}$. As the plasticity is independent of the time parameterization in statics, for convenience, proposing: $\Delta t_j = 1$. Considering m time step from t_0 to t_m and enforcing the yield condition only at the beginning and the end of the step, it has to minimize the objective function:

$$\bar{\Pi}(\boldsymbol{g}_0, \cdots, \boldsymbol{g}_m, \boldsymbol{q}_0, \cdots, \boldsymbol{q}_m, \dot{\boldsymbol{\lambda}}_0, \cdots, \dot{\boldsymbol{\lambda}}_m) = \sum_{j=1}^{j=m} (\boldsymbol{\Lambda}^T \dot{\boldsymbol{\lambda}}_j - \Delta \boldsymbol{q}_j^T \boldsymbol{G}\,\boldsymbol{g}_j + \Delta \boldsymbol{g}_j^T \boldsymbol{F}\,\boldsymbol{g}_j) \tag{18}$$

under the constrains of :

- Equilibrium (on the boundary, at each time step) :

$$g_{0,j} = -p(t_j), \quad g_{2\,(n+1)-1,j} = 0$$

- Plasticity (at every integration point g of every element e and at every time step) :

$$\boldsymbol{N}_{Y,e}^T(r_g)\boldsymbol{g}_j - \sigma_Y \leq 0, \quad \dot{\lambda}_{g,j} \geq 0, \quad \boldsymbol{N}_Y(r_g)\lambda_{g,j} = \boldsymbol{B}_e(r_g)\,\boldsymbol{L}_e \Delta \boldsymbol{q}_j - \boldsymbol{S}\,\boldsymbol{T}_e(r_g)\,\boldsymbol{M}_e \Delta \boldsymbol{g}_j$$

- Initial conditions :

$$\boldsymbol{g}_0 = \boldsymbol{0}, \qquad \boldsymbol{q}_0 = \boldsymbol{0}, \qquad \dot{\boldsymbol{\lambda}}_0 = \boldsymbol{0}$$

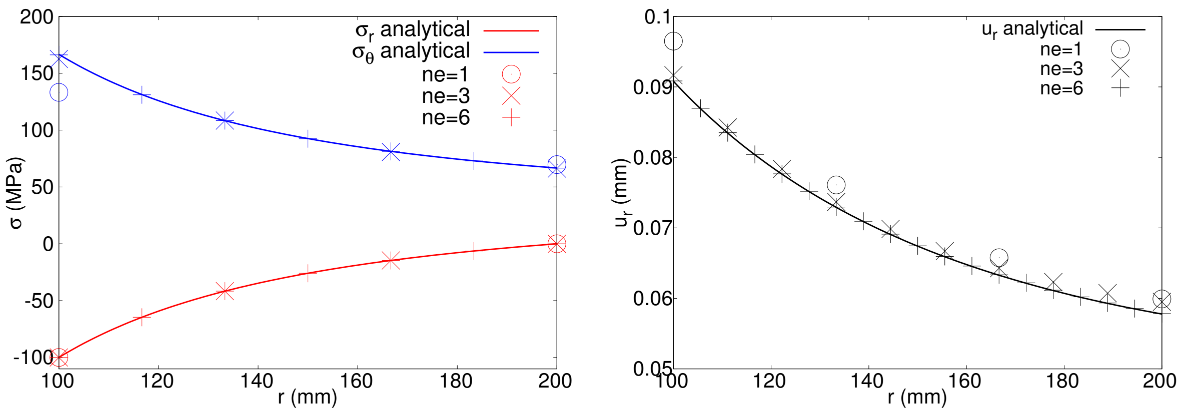
8 SIMULATION RESULTS

The thick tube has an internal radius a = 100 mm and external one b = 200 mm. A perfect plasticity material with Young modulus E = 210000 MPa, Poisson's ratio $\nu = 0.3$ and yield stress σ_Y = 360 MPa is chosen. Program is coded with *Matlab*, solver `fminsearch` is used to find the local minimum of functional with values of starting points are always 0.1 for all local values. In simulation, the pressure history $t \mapsto p(t)$ is imposed firstly in statics, the number of elements and time steps are given. By using the results of pre-processing, local minimum of the functional (18) is calculated.

In here, the thick tube is molded by one, three and six elements (ne = 1, 3, 6) with two time steps, the first time step is initial conditions for both elastic and elasto-plastic regimes.

8.1 Elastic regime

In elastic regime, numerical solutions are compared to analytical solution (Fig. 1). Imposed pressure is 100 MPa which is smaller than elastic limit.



(a) Stress (b) Radical displacement

Figure 1: Comparison of local values in elastic regime

With only one time step, stress and radical displacement field converge to analytical solution by increasing number of elements. BEN method works well in elastic regime.

8.2 Elasto-plastic regime

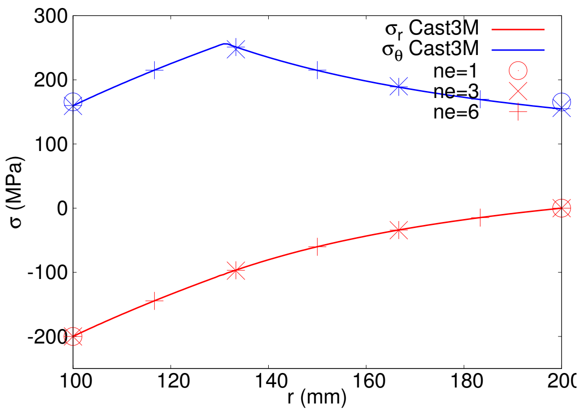
In this regime, numerical solutions of `Cast3M` [5] are reference solutions (Fig. 2) because analytical one not exists anymore. Imposed pressure is 200 MPa which is bigger than elastic limit but smaller than limit charge.

Like in elastic regime, with one time step, stress, radical displacement and plastic multiplier field converge to reference solution by increasing number of element. BEN method works well in elasto-plastic regime.

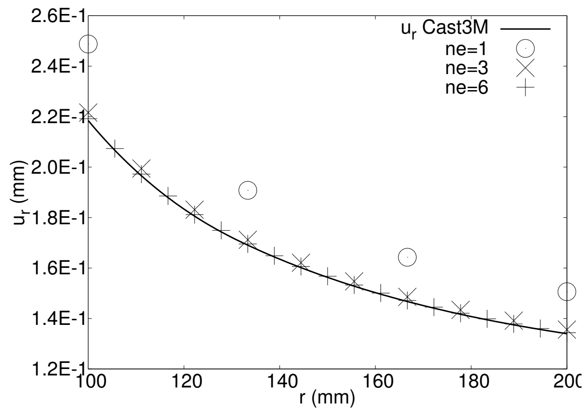
9 CONCLUSIONS AND FUTURE WORKS

The symplectic Brezis-Eleland-Nayroles principle makes it possible to have a coherent view of the global evolution by calculating all the steps simultaneously. With the present results, it can prove that this method is robust.

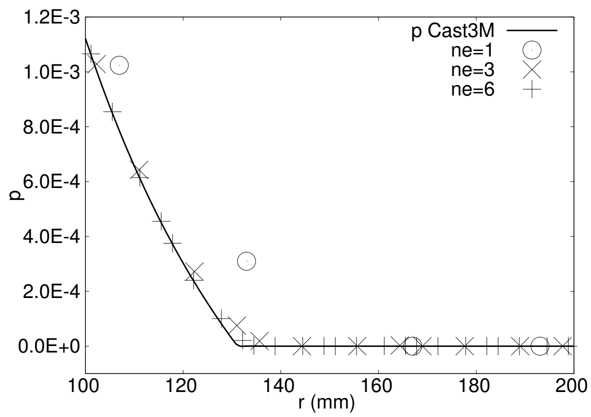
Next step is to test this method in dynamic case with a cyclic charge. Moreover, as this principle is space-time, the resolution is more expensive, hence the idea of combining it with a model reduction method such as the PGD (Proper Generalized Decomposition).



(a) Stress



(b) Radical displacement



(c) Plastic multiplier

Figure 2: Comparison of local values in elasto-plastic regime

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