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BAYESIAN UPDATING OF CONCRETE GRAVITY DAMS MODEL PARAMETERS USING STATIC MEASUREMENTS

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Abstract. Dams are fundamental infrastructures for energy production, flood control and agricultural-industrial sustenance. Most of them were built before the introduction of seismic regulations with no concerns about their dynamic behaviour. Nevertheless, in recent years, the international scientific community has been paying close attention to the seismic risk of existing dams.

Concrete gravity dams have never failed during earthquakes, so no case studies are available and their seismic behaviour can only be explored and investigated using numerical approaches. For this reason, finite element models must be calibrated through reliable procedures to obtain a sensible result. In this scenario, measurements acquired by a monitoring system and data from in-situ tests have taken on a major role as important sources of information.

Methods usually employed for this purpose require a low computational burden, however they are characterised by a high level of uncertainty. Probabilistic methods may be suitable to solve this inverse problem, but they always require considerable computing power, due to the high number of analyses needed, especially when stochastic finite elements are involved. In this paper, a procedure for the model parameters calibration in a Bayesian context is proposed. The novelty of this study is the use of a proxy model replicating the mechanical behaviour of a dam, in order to reduce the computational burden. This approach also allows us to estimate the global model error.

Two models, a single monolith and a complete 3D model of a large Italian dam, have been considered. After comparing the errors of different approaches, the best model simulating the observed behaviour of the dam was selected. The efficiency of the proposed methodology is also evaluated.

1 INTRODUCTION

Concrete gravity dams are a key component of the worldwide energy production system, but they are also used to control floods, for industrial purposes and more. Nowadays for environmental reasons only few new dams are being built in developed countries. A large part of the existing ones have been designed before the introduction of seismic regulation or have been built in those regions classified as seismic in a later time [1] and now they are ageing out fast. Therefore, in the recent years the scientific community has been paying more attention on the seismic assessment of dams, by evaluating their dynamic behaviour via numerical models [2].
In this scenario, reliable numerical models are fundamental and all available information about the structure must be used to reduce uncertainties. Two important sources in this regard are in situ tests and data recorded by the monitoring system during normal operation of the dam. The static monitoring system of concrete gravity dams usually acquires displacements of some significant points of the structure, together with environmental data as water level and air and water temperatures [4]. Several different methods to calibrate numerical model parameters are available in literature [3].
This paper proposes a procedure defined in a Bayesian framework which represents one of the first applications to concrete dams. The novelty lies in the approximation of the model response via a proxy model obtained through the general Polynomial Chaos Expansion technique (gPCE). For the sake of simplicity, in this first application only displacements due to basin level variations have been considered, by extracting them from total data as indicated in literature [3].
In paragraph 2 the probabilistic model is introduced for the general case and for concrete dams; in paragraph 3, the Bayesian updating is shown with focus on the likelihood function, on the definition of prior and proposal distributions and on the numerical algorithm; in paragraph 4 a brief review of the general polynomial chaos expansion is reported: in paragraph 5 the procedure is applied to the case of an Italian concrete gravity dam.

2 PROBABILISTIC MODEL

In this section, the probabilistic model for the FE model parameters updating of concrete dams is presented. The main sources of uncertainties are due to the measurement system, the deterministic model, the proxy model and the data pre-processing step. Following Gardoni [5], an additional set of explanatory functions accounts for the bias related to the other phenomena which are not included in the model. Generally, Bayesian updating procedures are computationally expensive, becoming prohibitive in conjunction with FE models with a large number of degrees of freedom. For this reason, in the present work a proxy model [6] has been created to reduce the computational burden, as it will be explained in detail in the paragraph 4.

2.1 General Formulation

In the present work a probabilistic additive model has been used to link the output of the deterministic model with recorded data. Let $\mathbf{x}$ the vector of the uncertain parameters of the deterministic model, the q-variate probabilistic model can be written as follows

$$C_{k,i}(\mathbf{x},\boldsymbol{\theta}_k,\boldsymbol{\Sigma}) = \hat{c}_{k,i}(\mathbf{x}) + \gamma_{k,i}(\mathbf{x},\boldsymbol{\theta}_k) + \sigma_k\epsilon_{k,i} \quad k = 1, ..., q \tag{1}$$

where $\gamma_k = \sum_{n=1}^{N} \theta_{n,k} h_{n,k}$ is the k-th correction term, that is a combination of the N explanatory functions $h_{n,k}$ via the combination coefficients collected in the vector $\boldsymbol{\theta}_k$; $C_{k,i}(\mathbf{x},\boldsymbol{\theta}_k,\boldsymbol{\Sigma})$ is either the i-th value of the k-th measure recorded by the monitoring system, or its transformation; $\hat{c}_{k,i}(\mathbf{x})$ is either the i-th value of the k-th response of the deterministic model, or its transformation; $\epsilon_{k,i}$ are normal random variables with zero mean and unit variance, σ_k is the k-th standard deviation of the probabilistic model error and $\boldsymbol{\Sigma}$ is the covariance matrix of the random variables $\sigma_k\epsilon_k$. The additive corrected model is valid when the following assumptions are satisfied:

(a) the model standard deviation is independent of x (homoskedasticity assumption) and

(b) the model error has the normal distribution (normality assumption).

Sometimes, in order to satisfy the homoskedasticity assumption, the reference parameter must be transformed using the family functions reported in Yang [7]; in this case a logarithmic function has been used.

2.2 Application of the probabilistic model to concrete gravity dams

The aim of this work is the updating of the materials mechanical parameters, by means of data recorded by static monitoring system. Materials have been assumed linear elastic, since static displacements are really small in comparison to the characteristic dimension of the system and the stress level is extremely limited, well below the material strength. The mechanical behaviour of elastic materials is completely described by the fourth-order constitutive matrix $\mathbb{C}$ [8], whose components may be treated as random variables in a Bayesian updating procedure.

In this work, the reference measure is the displacement of a control point recorded only in the upstream/downstream direction. If the displacement data were available both upstream-downstream and in the cross-valley direction, it might be worthwhile to update the parameters of an orthotropic material through a multivariate probabilistic model. Conversely, when only upstream-downstream measurements are available, as in the present case, mechanical parameters along the cross-valley have little influence on the structural deformability in the upstream-downstream direction, so the assumption of isotropic material is the best choice. Thus the matrix $\mathbb{C}$ becomes as in equation (2).

Uncertainties parametrization for elastic materials by selecting the bulk modulus K and the shear modulus G as random variables is a particularly convenient choice, since these are physically and statistically independent. By choosing log-normal distributions for K and G, the elastic constitutive matrix is positive defined. In this paper, K and G of both concrete and foundation soil have been treated as random variables and collected in the $\mathbf{x}$ vector. Furthermore, since the displacement of a control point in one direction only is considered, a uni-variate form of the probabilistic model can be adopted.

$$\mathbb{C} = \begin{bmatrix} K+4G/3 & K-2G/3 & K-2G/3 & 0 & 0 & 0 \\ K-2G/3 & K+4G/3 & K-2G/3 & 0 & 0 & 0 \\ K-2G/3 & K-2G/3 & K+4G/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \tag{2}$$

Let $\delta_i^Q(\mathbf{x})$ the i-th value of the recorded displacement related to the basin level variation and $\hat{\delta}_i^Q(\mathbf{x})$ the i-th value of the reference displacement obtained as deterministic model output, and using logarithmic transformation functions to satisfy the homoskedasticity assumption, the probabilistic model can be written as follows

$$\ln\left(\delta_i^Q(\mathbf{x},\boldsymbol{\theta},\sigma)\right) = \ln\left(\hat{\delta}_i^Q(\mathbf{x})\right) + \gamma_i(\mathbf{x},\boldsymbol{\theta}) + \sigma\epsilon \tag{3}$$

In this special case, only one explanatory function $h_0 = 1$ has been introduced to capture the potential bias representing the discrepancy between the measured data reference system and the one of the FE model displacements.

3 BAYESIAN UPDATING VIA MONTE CARLO MARKOV CHAIN (MCMC)

All the uncertain parameters are collected in the vector $\boldsymbol{\Theta} = (\mathbf{x},\boldsymbol{\theta},\sigma)^T$, where $\mathbf{x}$ contains the unknown material parameters, σ is the global error standard deviation and $\boldsymbol{\theta}$ is the combination coefficients vector. The prior state of knowledge about $\boldsymbol{\Theta}$, called prior distribution $p(\boldsymbol{\Theta})$, is updated via the well-known Bayes rule [9], using the new information $\mathbf{y}$ about the system. The posterior distribution $p(\boldsymbol{\Theta}|\mathbf{y})$, which represents the updated state of knowledge about the parameters $\boldsymbol{\Theta}$, is obtained as follows

$$p(\boldsymbol{\Theta}|\mathbf{y}) = \kappa L(\boldsymbol{\Theta}|\mathbf{y})p(\boldsymbol{\Theta}) \tag{4}$$

where $\kappa = \left[\int L(\boldsymbol{\Theta}|\mathbf{y})p(\boldsymbol{\Theta})d\boldsymbol{\Theta}\right]^{-1}$ is the normalizing factor and $L(\boldsymbol{\Theta}|\mathbf{y})$ is the likelihood function. Since the global error is normally distributed, the likelihood function can be defined as follows [10, 11]

$$L(\mathbf{x},\boldsymbol{\theta},\sigma) \propto \prod_{i=1}^{l}\left\{\frac{1}{\sigma}\varphi\left[\frac{r_i(\mathbf{x},\boldsymbol{\theta})}{\sigma}\right]\right\} \tag{5}$$

where $\varphi(\bullet)$ is the standard normal density function and $r_i(\mathbf{x},\boldsymbol{\theta})$ is the i-th residual

$$r_i(\mathbf{x},\boldsymbol{\theta}) = \ln\left(\delta_i^Q(\mathbf{x},\boldsymbol{\theta},\sigma)\right) - \ln\left(\hat{\delta}_i^Q(\mathbf{x})\right) - \gamma_i(\mathbf{x},\boldsymbol{\theta}) \tag{6}$$

Once the probabilistic model is defined, the choice of the solution procedure is fundamental. In real problems a closed-form solution cannot usually be performed, then numerical procedures are needed. Several different numerical approaches for the determination of the posterior distribution are available. In this work Monte Carlo Markov Chain (MCMC)

with Metropolis-Hastings algorithm [12] has been implemented. This particular approach is powerful and versatile because it can draw samples from any target probability density function π for the uncertain parameters $\mathbf{\Theta}$, since it only requires the pdf can be calculated at $\mathbf{\Theta}$. Moreover, the new proposed parameter samples $\mathbf{\Theta}^*$ are generated by a proposal density function $q\left(\mathbf{\Theta}_t, \mathbf{\Theta}^*\right)$, depending on the current state of the chain $\mathbf{\Theta}_t$. The proposal $\mathbf{\Theta}^*$ can be accepted as next state of the chain $\mathbf{\Theta}_{t+1} = \mathbf{\Theta}^*$ with acceptance probability $\alpha\left(\mathbf{\Theta}_t, \mathbf{\Theta}^*\right)$, or can be rejected otherwise. The specification of the probability α allows generating a Markov chain with desired target density π.
On the other hand, MCMC requires a large analyses number to reach convergence and, for this reason its application with FE models could be prohibitive. In this work, a proxy model based on the gPCE has been used in order to reduce the computational burden [13]. Nevertheless, the proxy model introduces an additional error (included in the global error of the equation 3), due to the differences between the FE model output and the proxy model response.

3.1 Prior distributions

Prior distributions represent the initial state of knowledge about the uncertain parameters. Their definition is a fundamental step in the updating procedure, particularly when few new information is available. Usually, in the case of dams, results of survey campaigns are available, allowing a precise definition of prior distributions of the materials mechanical parameters, i.e. K and G. When no information are available, as in the case of the combination coefficients collected in $\boldsymbol{\theta}$ and the global error standard deviation σ, non-informative prior distributions have to be used, to avoid affecting updating result. Following Gardoni [10], non-informative priors can be written as follows

$$p(\sigma) \propto \frac{1}{\sigma} \tag{7}$$

$$p(\boldsymbol{\theta}) \propto \frac{\theta_n}{\sigma} \tag{8}$$

Thus, in this case, given the large amount of observed data, any reasonable choice of the prior has little influence on the posterior estimates of the mechanical parameters.

3.2 Convergence diagnostic

Convergence diagnostic is used to determine whether the samples generated by MCMC are representative of the underlying equilibrium distribution. In this paper, the commonly used diagnostics metric proposed by Brooks and Gelman [14] has been adopted. The convergence of Markov chain simulation is reached when inferences for quantities of interest do not depend on the starting point. Monitoring convergence is obtained by comparing several inferences performed with different starting points. The diagnostics metric is based on the calculation of the Multivariate Potential Scale Reduction Factor (MPSRF) $\hat{R}^p$ in the multivariate case shown as follows

$$\hat{R}^p = \max_{\mathbf{a}} \frac{\mathbf{a}^T \hat{\mathbf{V}} \mathbf{a}}{\mathbf{a}^T \mathbf{W} \mathbf{a}} \tag{9}$$

where $\hat{\mathbf{V}}$ is the total variance extended to the multivariate case, $\mathbf{W}$ is the within-sequence variance extended to the multivariate case and $\mathbf{a}$ is a vector used to achieve the maximum value of the ratio.
In general, MPSRF is defined as the ratio between total variance and within-sequence variance. It represents the upper bound of the maximum of the univariate Potential Scale Reduction Factor (PSRF) statistics $\hat{R}$ among variables. When the convergence is reached, the between-sequence variance should be negligible, obtaining $\hat{R}^p = 1$. Usually, $\hat{R}^p = 1.1$ is considered as acceptable, but when the dimension of the problem increases, a convergence criterion $\hat{R}^p = 1.5$ is allowed [14].

4 GENERAL POLYNOMIAL CHAOS EXPANSION

The general Polynomial Chaos Expansion (gPCE) is an uncertainties propagation technique through a deterministic model, which also allows building a proxy model, called response surface [6], which can be used in the updating procedure.
The uncertain structural response $u(\mathbf{x})$, which is a function of the set of unknown parameters collected in $\mathbf{x}$, can be described in a probabilistic space defined by the triplet $(\Omega, \mathfrak{F}, \mathbb{P})$: where Ω is the space of all events, $\mathfrak{F}$ is the σ-algebra and $\mathbb{P}$ the probability measure. Assuming that $u(\mathbf{x})$ is smooth enough to be represented in terms of some simple random variables $\zeta(\mathbf{x})$ (e.g. Gaussians, uniform, etc.), via the PCE [15] the structural response can be approximated by $u_N(\zeta(\mathbf{x}))$, defined as follows

$$u(\zeta(\mathbf{x})) \approx u_N(\zeta(\mathbf{x})) = \sum_{\alpha \in \mathbf{I}} u^{(\alpha)} \mathbf{\Psi}_\alpha(\zeta(\mathbf{x})) \tag{10}$$

where $\mathbf{\Psi}_\alpha(\zeta(\mathbf{x}))$ represents the multivariate orthogonal polynomials with finite multi-index set $\mathbf{I}$ and $u^{(\alpha)}$ are the polynomial coefficients. Exploiting the orthogonality condition, all statistics can be retrieved from gPCE in a straightforward manner, as in the case of the expected values, defined as follows

$$\mathbb{E}\left[u(\zeta(\mathbf{x}))\right] \approx \mathbb{E}\left[u_N(\zeta(\mathbf{x}))\right] = \int \sum_{\alpha \in \mathbf{I}} u^{(\alpha)} \mathbf{\Psi}_\alpha(\zeta(\mathbf{x})) dF_{\zeta(\mathbf{x})} \zeta(\mathbf{x}) = u^0 \tag{11}$$

The polynomial coefficients $u^{(\alpha)}$ are determined based on the deterministic model solutions, e.g. FE models. Several different approaches are available to solve this task. In this work a regression procedure which minimize the error between the gPCE and the model response has been used. The choice of maximum polynomial degree and analyses number must consider both the accuracy of the result and the computational burden. Moreover, the choice of the polynomial family depends on the probability distributions of the unknown parameters, as accurately indicated in Xiu [6].
Finally, the model sensitivity analysis can be performed very easily using the gPCE. In this work Sobols coefficients [16] allow determining the influence of each model parameter on the final results.

5 CASE OF STUDY

The procedure has been applied to the case of an Italian gravity dam, composed by 11 monoliths separated with vertical contraction joints. Its maximum height is 108 m and it has a curved shape in plan, with curvature radius of 150 m and crest length of 234.25 m. The proposed model makes use of static displacements due to basin level variations during dam normal operation. To this aim, displacements have been previously processed to eliminate their thermal variations related part.
Two models have been set up: a complete 3D model with bonded monoliths and a 3D single monolith model, representing only the central spillway monolith. Both models shown in Figure 1 have been created in ANSYS r17 [17]. The complete 3D model, whose mesh is composed of 29981 second order Hex-dominant elements, is restrained along the three directions at the base and along the horizontal directions on the sides. The single monolith model is made of 3587 second order Hex-dominant elements and is laterally unrestrained. The reference parameter is the displacement of a control point placed in the upper part of the central spillway monolith, where displacements recorded by the inverse pendulum are available. The unknown model parameters are the elastic moduli of the concrete K_{cls} and G_{cls} and those of the foundation soil K_{soil} and G_{soil}. The prior distributions of the model parameters have been defined basing on in-situ tests results. The global error standard deviation σ and the combination coefficients of the explanatory functions $\boldsymbol{\theta}$ have been treated as random variables and updated. Since no information about σ and $\boldsymbol{\theta}$ are available, their prior distributions have been defined as non-informative. The prior distributions statistics of the mechanical parameters are shown in table 1.

Table 1: Prior distributions of the mechanical parameters

	K_{cls} [MPa]	G_{cls} [MPa]	K_{soil} [MPa]	G_{soil} [MPa]
distributions	LN	LN	LN	LN
mean values	12148	8364.2	41000	23428
standard deviations	4915.1	3384.2	9393.1	5367.4

Basin levels data and control point displacements recorded during normal operation of the dam are the information available to update the probabilistic model. Dam displacements are previously processed according to De Sortis [3], in order to separate thermal and basin level related displacement contributions. The latter are shown in figure 2.
The first step of the procedure is the composition of the proxy models; the maximum polynomial degree has been set equal to 3 and a complete basis has been built requiring 1024 analyses for each model. The sensitivity analysis of the proxy model provided the Sobols coefficients of the two models, as shown in figure 3. Sobols coefficients related to concrete and soil parameters of the two models must be analysed in comparative terms. Their values are so low in comparison to the Sobols coefficient of the basin level variation which is not shown here. Anyway, in both models the most significant parameters are those of the concrete.

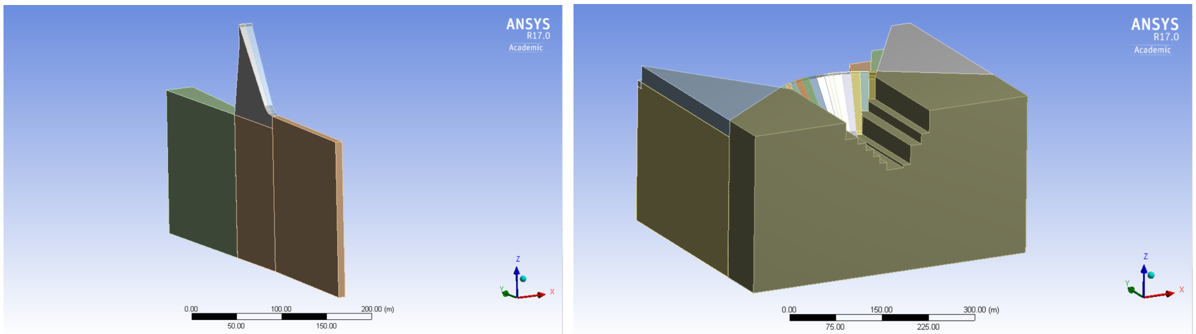

Figure 1: Single monolith model and 3D complete model

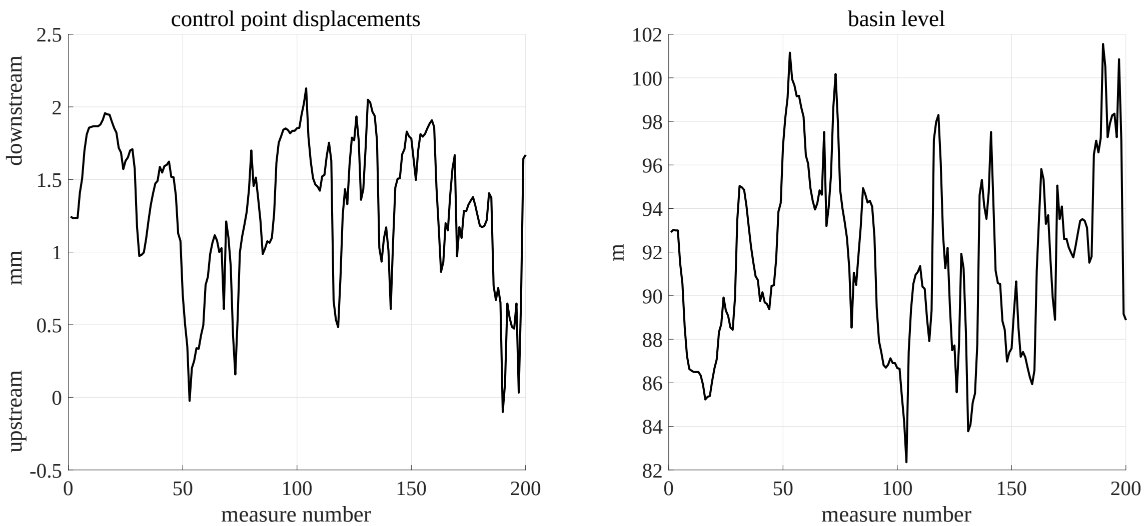

Figure 2: Control point displacements (left) and basin level (right)

Table 2: Posterior distributions of the 3D complete model parameters

	K_{cls} [MPa]	G_{cls} [MPa]	K_{soil} [MPa]	G_{soil} [MPa]	σ
mean values	12149	8653.9	40769	22517	0.0332
standard deviations	220.17	152.05	553.10	169.53	0.0016

Table 3: Posterior distributions of the single monolith model parameters

	K_{cls} [MPa]	G_{cls} [MPa]	K_{soil} [MPa]	G_{soil} [MPa]	σ
mean values	36264	23545	40224	22571	0.0661
standard deviations	8436.5	5152.5	3032.3	958.29	0.0984

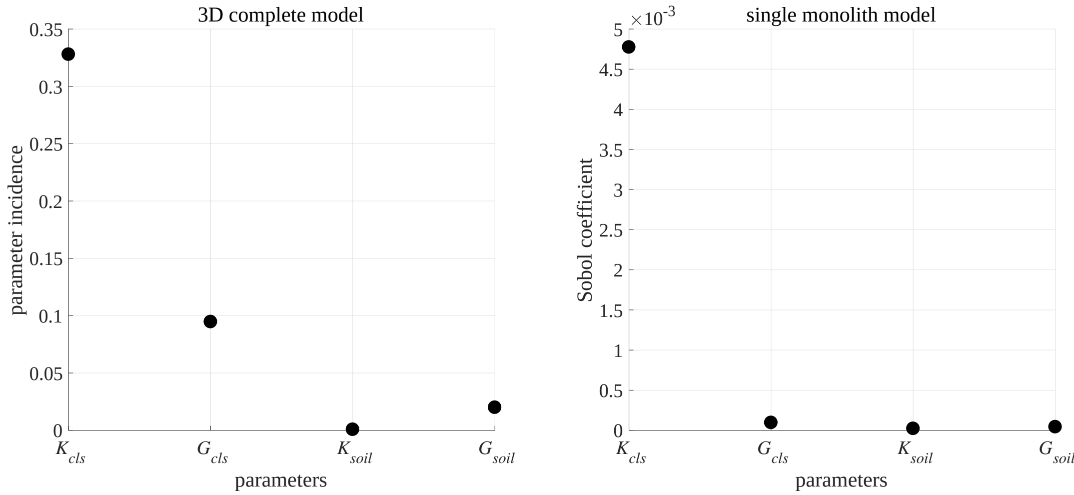

Figure 3: Sobols coefficients: 3D complete model and single monolith model

Unknown parameters have been updated using the proxy models and the results in terms of mean and variance values are reported in table 2 and 3. In figure 4 the comparison between prior and posterior distributions is shown and, in figure 5, the effects of the model parameters updating are shown in terms of residuals.

The convergence of the MCMC algorithm has been checked according to the paragraph 4.2, considering $\hat{R}^p = 1.5$. As for the single monolith model, updated values of the mean and are higher than both the complete model and the prior distribution. The reason lies in the lower stiffness of the single monolith in respect to that of the complete model, due to different lateral boundary conditions. Moreover, standard deviations of the single monolith model parameters are higher than those of the complete model. This fact is also evident in terms of mean values, that in the single monolith model is twice the complete model. The good agreement between recorded and calculated results can be observed in figure 5, where the residuals in both deterministic and updated cases are shown. The proximity of the points to the bisector line indicates that the model control point displacements are close to the recorded ones.

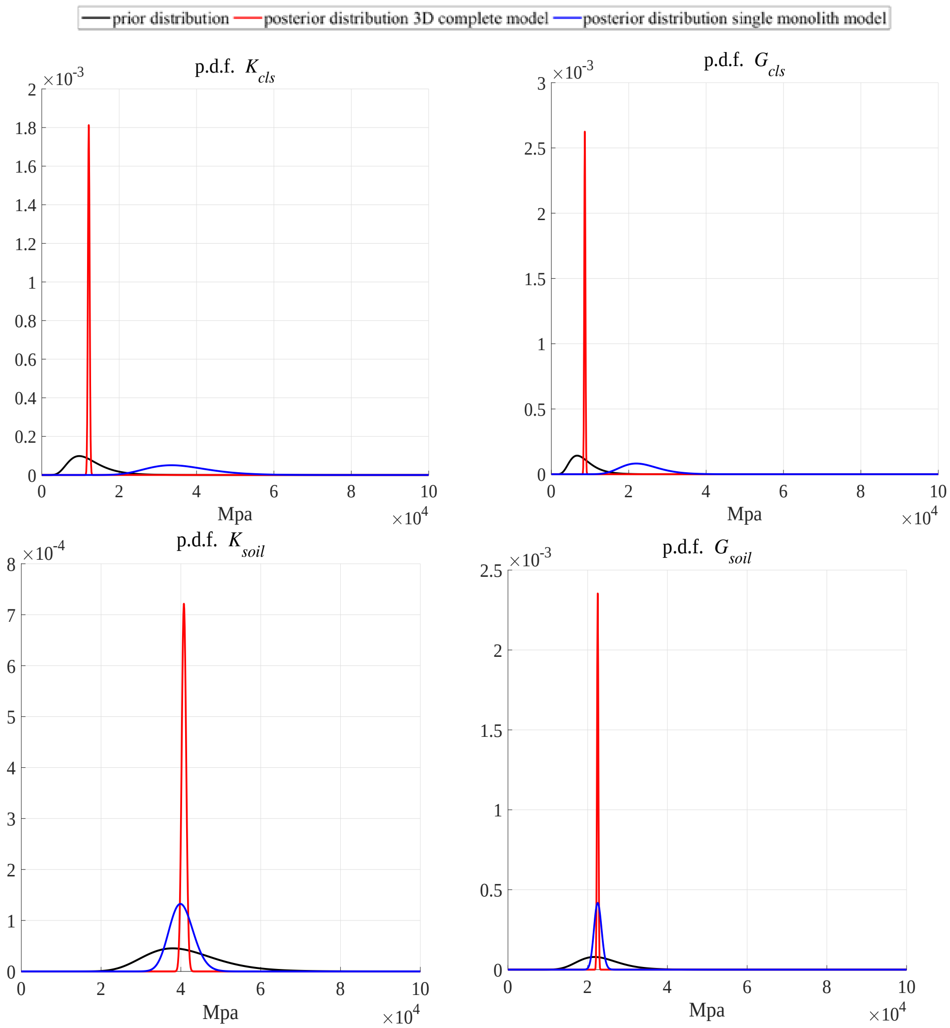

Figure 4: Comparison between prior and posterior distributions

6 CONCLUSIONS

In the present work, a procedure to update the FE model parameters of concrete dams has been set up in a probabilistic framework. Displacements recorded by the control system and in-situ tests results have been used to estimate model parameters values which provide a model response as similar as possible to the real structural behaviour. The additive corrected probabilistic model allows considering the global error and its standard deviation, which can be used to compare different classes of models. Moreover, the sets of explanatory functions allow correcting the probabilistic model, reducing bias related to unknown phenomena. The FE model response has been approximated by a proxy model, reducing the computational burden, via the general polynomial chaos expansion. This technique is particularly advantageous because it allows solving the forward problem in straightway manner and performing a sensitivity analysis on the results. The sensitivity analysis is a fundamental step because it allows determining the most influencing parame-

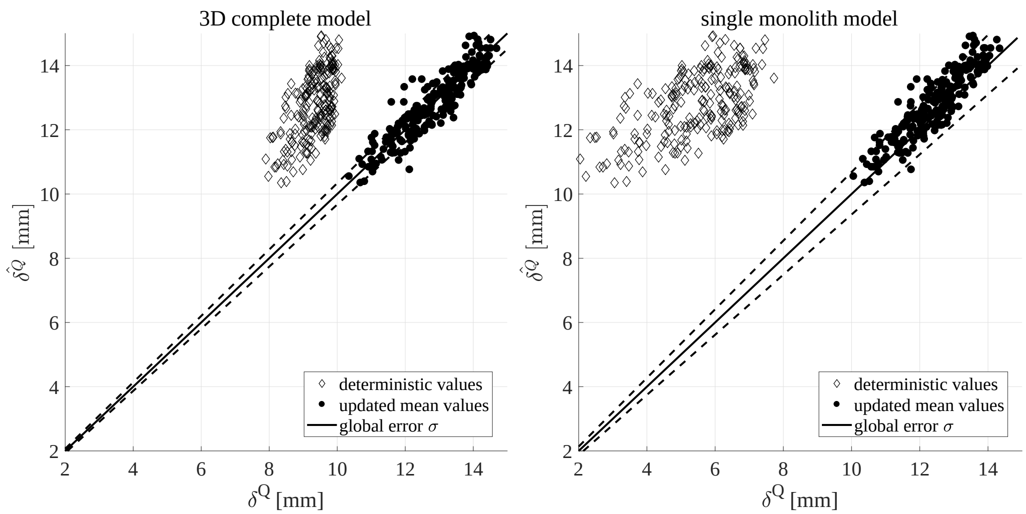

Figure 5: Effects of the model parameters updating on the residuals

ters. Finally, the application of the presented procedure to a concrete Italian gravity dam shows the effectiveness of the method and its feasibility in real cases application. Future developments concern the introduction of thermal displacements within the probabilistic model, eliminating the pre-process data step. The presented procedure could be the base for a better estimation of the structural reliability and it can be regarded as a starting point of health monitoring system application for dams.

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