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NECESSARY OPTIMALITY CONDITIONS FOR OPTIMALLY CONTROLLED DISSIPATIVE MECHANICAL SYSTEMS MODELLED THROUGH FRACTIONAL DERIVATIVES

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Abstract. Employing a phase space which includes the (Riemann-Liouville) fractional derivative of curves evolving on real space [5], we develop a restricted variational principle for Hamiltonian systems yielding the so-called restricted fractional Hamilton equations. Moreover, we introduce controlled external forces in the dynamics by means of an extension of the Lagrange-d'Alembert principle. In this scenario, we establish the Fractional Optimal Control Problem, whose necessary optimality conditions we obtain by using calculus of variations. We observe that these optimality conditions are a system of algebraic-differential equations with fractional terms and mixed (initial-final) endpoint conditions. We treat the example of a linearly damped harmonic oscillator and minimum effort with external forces which are linear in the control variables. The deviation of the expected behaviour, with respect to the non-fractional case, is explained in terms of the geometry of a higher order cotangent bundle.

1 INTRODUCTION

Optimal control theory and variational mechanics ([7]) have their common origin in the calculus of variations. Our objective is to obtain optimality conditions (also known as Pontryagin's maximum principle [8] assuming enough regularity) for linearly damped mechanical systems subject to external controlled forces as the Euler-Lagrange equations (via Hamilton's principle [4]) of a particular augmented action functional. In order to model the dissipative forces we employ the recent approach by the authors [5] based on previous literature ([1, 2, 3, 9]), i.e. a restricted variational principle defined on a $\alpha-$fractional phase space $\mathcal{T}\mathbb{R}^d$ and curves evolving on real space which, in the case of $\alpha = 1/2$ and mechanical Lagrangian functions, model linear damping. The obtained dynamical equations are so-called *restricted fractional Euler-Lagrange equations*.

In the present paper, we take the dual approach. Namely, we define the dual of the above mentioned $\alpha-$fractional phase space, say $\mathcal{T}^*\mathbb{R}^d$. This space, involves the momenta and the dual variables of the fractional derivatives, all of them defined through the *fractional Legendre transform*. The fractional Legendre transform is established as the fiber derivative of the Lagrangian, and provides as well a Hamiltonian function on $\mathcal{T}^*\mathbb{R}^d$. Through the Hamilton's principle we obtain the so-called *restricted fractional Hamilton equations*, which become the usual Hamilton equations for linearly damped mechanical systems when $\alpha = 1/2$. The controlled external forces are easily introduced in the new fractional Hamiltonian dynamics by means of an extension of the Lagrange-d'Alembert principle. Then, we extend the usual notion of *Optimal Control Problem*, considering the new restricted fractional Hamilton equations (for a general α) with controlled external forces as constraints. This allows to define the augmented action functional in the usual way, leading to the optimality conditions via calculus of variations.

Outline. In §2 we provide all the background material. In §3 we display the *restricted fractional Euler-Lagrange equations* (13) in [5] and obtain their Hamiltonian version, i.e. the *restricted fractional Hamiltonian equations* (**Theorem 3.2**). In **Corollary 3.3** we show that the restricted fractional Hamiltonian equations (17) become the usual Hamiltonian equations for linearly damped mechanical systems when $\alpha = 1/2$. Furthermore, we introduce the controlled external forces in the fractional dynamics through the extension of the Lagrange-d'Alembert principle described in **Theorem 3.5**. In §4 we extend the usual notion of an Optimal Control Problem, allowing the forced fractional dynamics. Also, we derive the necessary optimality conditions through calculus of variations. Finally, we treat the example of a linearly damped harmonic oscillator and minimum effort with external forces which are linear in the control variables.

2 PRELIMINARIES

2.1 Optimal Control Problem

We shall define the Optimal Control Problem (OCP henceforth) in the Hamiltonian fashion [6]. The evolution of a forced mechanical system evolving on a finite dimensional smooth manifold Q is given by smooth curves $q : [a, b] \subset \mathbb{R} \to Q$ determined through the usual Hamiltonian dynamics provided by the Lagrange-d'Alembert principle[1], i.e.

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} + f_H(q, p, u), \tag{1}$$

where $H : T^*Q \to \mathbb{R}$ is the Hamiltonian function defined on T^*Q, the cotangent bundle of Q with local coordinates (q, p). On the other hand, $f_H : T^*Q \times U \to T^*Q$ represents the controlled external forces, where $U \equiv \mathbb{R}^m$ is the control space, with $m \leq \mathrm{dim}Q$. Given

[1] The Lagrange-d'Alembert principle in its Hamiltonian version is established as

$$\delta \int_a^b \{\langle p, \dot{q}\rangle - H(q, p)\}\, dt + \int_a^b \langle f_H(q, p, u), \delta q\rangle\, dt = 0.$$

a *cost functional*

$$J(q,p,u) = \int_a^b C(q(t),p(t),u(t))dt + \Phi(q(b),p(b)),$$

where $C : T^*Q \times U \to \mathbb{R}$ and $\Phi : T^*Q \to \mathbb{R}$ (Mayer term) are continuously differentiable functions, we define the OCP as follows:

Problem 2.1 (Optimal Control Problem)

$$\begin{aligned}
\min_{(q,p,u)} J(q,p,u) &= \int_a^b C(q(t),p(t),u(t))dt + \Phi(q(b),p(b)), && \text{(2a)}\\
\text{subject to} \qquad \dot q(t) &= \ \ \partial H(q(t),p(t))/\partial p, && \text{(2b)}\\
\dot p(t) &= -\partial H(q(t),p(t))/\partial q + f_H(q(t),p(t),u(t)), && \text{(2c)}\\
(q(a),p(a)) &= (q^a,p^a). && \text{(2d)}
\end{aligned}$$

According to this, since the final time b is fixed, we are establishing a fixed-time, free-endpoint problem. Pontryagin's maximum principle [8] provides necessary conditions for optimality of feasible trajectories $\eta(\cdot) = (q(\cdot),p(\cdot),u(\cdot))$ for Problem 2.1 (which are curves satisfying (2b)-(2d); they are optimal if moreover they satisfy (2a)). For regular systems (i.e. $C, \Phi, f_H, \partial H/\partial q$ and $\partial H/\partial q$ are differientiable w.r.t. q, p and u and $u(\cdot) \in C^0$, $\dot q(\cdot) \in C^1$ and $q(\cdot) \in C^2$ [7]) and when there are no further constraints on the control variables and the final state, these conditions can be derived by means of the Euler-Lagrange equations of the augmented cost functional:

$$\begin{aligned}
\mathcal{S}(\eta,\lambda) = \int_a^b \Big\{ & C(q(t),p(t),u(t)) + \langle \lambda^q(t), \dot q(t) - \partial H(q(t),p(t))/\partial p\rangle \\
&+ \langle \lambda^p(t), \dot p(t) + \partial H(q(t),p(t))/\partial q - f_H(q(t),p(t),u(t))\rangle \Big\}\, dt + \Phi(q(b),p(b)),
\end{aligned} \tag{3}$$

where $\lambda(t) = (\lambda^q(t), \lambda^p(t))$ is called the *adjoint variable* or *costate* of the system and we consider all states and costates at final time b as independent variables. Under these conditions and considering separable Hamiltonians $H(q,p) = K(p) + P(q)$ for sake of simplicity, the necessary optimality conditions for feasible curves are

$$\begin{aligned}
\dot q &= \frac{\partial K}{\partial p}, & \dot p &= -\frac{\partial P}{\partial q} + f_H, && \text{(4a)}\\
\dot\lambda^q &= \lambda^p\left(\frac{\partial^2 P}{\partial q^2} - \frac{\partial f_H}{\partial q}\right) + \frac{\partial C}{\partial q}, & \dot\lambda^p &= -\lambda^q\,\frac{\partial^2 K}{\partial p^2} - \lambda^p\,\frac{\partial f_H}{\partial p} + \frac{\partial C}{\partial p}, && \text{(4b)}
\end{aligned}$$

$$0 = -\lambda^p\,\frac{\partial f_H}{\partial u} + \frac{\partial C}{\partial u}, \tag{4c}$$

$$q(a) = q^a,\ p(a) = p^a,\ \lambda^q(b) = -\frac{\partial \Phi}{\partial q}\Big|_b,\ \lambda^p(b) = -\frac{\partial\Phi}{\partial p}\Big|_b, \tag{4d}$$

which is a system of algebraic-differential equations with mixed (initial-final) endpoint conditions. From (4c) we observe that, if the matrix $\big[-\lambda^p\,(\partial^2 f_H/\partial u^2) + \partial^2 C/\partial u^2\big]$ is invertible, then according to the implicit function theorem we will be able to determine u as a function of the rest of variables, i.e. $u = u(q, p, \lambda^p)$. Inserting this into (4a)-(4b) we end up with a pure system of differential equations with mixed endpoint conditions where the control variables are absent.

2.2 Riemann-Liouville fractional derivatives

Let $\alpha \in [0, 1] \subset \mathbb{R}$ and $f : [a, b] \to \mathbb{R}$ a smooth function. The Riemann-Liouville fractional derivatives are defined by

$$D^\alpha_- f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} f(\tau) d\tau, \tag{5a}$$

$$D^\alpha_+ f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau - t)^{-\alpha} f(\tau) d\tau, \tag{5b}$$

for $t \in$[a,b], and $\Gamma(z)$ the gamma function ([10]). As it is well-known, the fractional derivatives are non-local operators: in the sequel $-$ and $+$ will denote the retarded and advanced cases, respectively. Let us consider two smooth functions f, g. The fractional integration by parts rule is given by

$$\int_a^b f(t) D^\alpha_\sigma g(t) dt = \int_a^b \left(D^\alpha_{-\sigma} f(t) \right) g(t) dt, \tag{6}$$

where σ stands either for $-$ or $+$. An important feature of fractional integrals is, when $\alpha = 1/2$:

$$D^{1/2}_- D^{1/2}_- f(t) = \frac{d}{dt} f(t), \;\; D^{1/2}_+ D^{1/2}_+ f(t) = -\frac{d}{dt} f(t). \tag{7}$$

See [10] for more details. According to the above definitions, the fractional derivatives are $\mathbb{R}$-valued. To see this, it is enough to note that $f(t)$ is $\mathbb{R}$-valued, as well as $(t-\tau)^{-\alpha}$ for $t > \tau$ and $(\tau - t)^{-\alpha}$ for $\tau > t$.

2.3 Fractional phase space

Henceforth we shall set $Q = \mathbb{R}^d$. Consider smooth curves $\gamma_x : [a, b] \subset \mathbb{R} \to \mathbb{R}^d$, $\gamma_y : [a, b] \subset \mathbb{R} \to \mathbb{R}^d$ for $d \in \mathbb{N}$, both belonging to $C^\infty([a, b], \mathbb{R}^d)$. Their local representation are given by $\gamma_x(t) = (x^1(t), ..., x^d(t))$, $\gamma_y(t) = (y^1(t), ..., y^d(t))$, $t \in [a, b]$. On the other hand, $\tilde{\gamma} : [a, b] \to \mathbb{R}^d \times \mathbb{R}^d$ is formed as $\tilde{\gamma} = (\gamma_x, \gamma_y)$.

With these ingredients, we can form the *double tangent bundle* and *double fractional tangent bundle*, $\mathbb{T}\mathbb{R}^d := ((\gamma_x, \gamma_y), (\dot{\gamma}_x, \dot{\gamma}_y))$ and $\mathbb{T}^\alpha\mathbb{R}^d := ((\gamma_x, \gamma_y), (D^\alpha_- \gamma_x, D^\alpha_+ \gamma_y))$, respectively. Their structure as vector bundles over $\mathbb{R}^d \times \mathbb{R}^d$ is explained in detail in [5].

Furthermore, with these two bundles we can construct the *fractional tangent phase space*:

$$\mathfrak{T}\mathbb{R}^d := \mathbb{T}\mathbb{R}^d \otimes_{\mathbb{R}^d \times \mathbb{R}^d} \mathbb{T}^\alpha\mathbb{R}^d. \tag{8}$$

Thus, $\mathcal{V}_{\tilde{\gamma}} := (\gamma_x, \gamma_y, \dot{\gamma}_x, \dot{\gamma}_y, D^\alpha_- \gamma_x, D^\alpha_+ \gamma_y) \in \mathfrak{T}\mathbb{R}^d$ is locally described by $\mathcal{V}_{\tilde{\gamma}} = (x, y, \dot{x}, \dot{y}, D^\alpha_- x,$ $D^\alpha_+ y)$, where we omit the $i = 1, ..., d$ superindex for simplicity. The bundle projection is given by $\mathcal{T}(\mathcal{V}_{\tilde{\gamma}}) = (x, y)$.

The construction of the dual bundle of (8), which we shall name *fractional cotangent phase space*

$$\mathfrak{T}^*\mathbb{R}^d := \mathbb{T}^*\mathbb{R}^d \otimes_{\mathbb{R}^d\times\mathbb{R}^d} \mathbb{T}^{\alpha*}\mathbb{R}^d,$$

follows straightforwardly from the dual bundles $\mathbb{T}^{\alpha*}\mathbb{R}^d$ and $\mathbb{T}^*\mathbb{R}^d$. For $\mathcal{A}_{\tilde{\gamma}} \in \mathfrak{T}^*\mathbb{R}^d$, we fix local coordinates

$$\mathcal{A}_{\tilde{\gamma}} = (x, y, p_x, p_y, p_x^\alpha, p_y^\alpha). \tag{9}$$

The bundle projection $\mathcal{P} : \mathfrak{T}^*\mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$ is locally given by $\mathcal{P}(\mathcal{A}_{\tilde{\gamma}}) = (x, y)$. Furthermore, the fiber $\mathcal{P}^{-1}(\mathbb{R}^d \times \mathbb{R}^d)$ is a vector space with dimension $4d$; and $\mathfrak{T}^*\mathbb{R}^d$ is locally the Cartesian product of 6 copies of $\mathbb{R}^d$ (equivalently to $\mathfrak{T}\mathbb{R}^d$, see [5] for more details).

3 RESTRICTED FRACTIONAL DYNAMICS

3.1 Restricted fractional Euler-Lagrange equations

Consider $C^\infty(x_a, x_b; \mathbb{R}^d \times \mathbb{R}^d)$ the set of curves $\tilde{\gamma} = (\gamma_x, \gamma_y)$ with fixed endpoints $\tilde{\gamma}(a) = (x_a, x_b)$, $\tilde{\gamma}(b) = (x_b, x_a)$. A *varied curve* of $\tilde{\gamma}$ is a map $\Gamma : \mathbb{R} \times [a, b] \to \mathbb{R}^d \times \mathbb{R}^d$, $\Gamma(\epsilon, t) := \tilde{\gamma}(t) + \epsilon\delta\tilde{\gamma}(t)$, defined such that the *variation* $\delta\tilde{\gamma}$ vanishes at the endpoints, i.e. $\delta\tilde{\gamma}(a) = \delta\tilde{\gamma}(b) = 0$. Observe that this implies

$$\delta x(a) = \delta y(a) = \delta x(b) = \delta y(b) = 0. \tag{10}$$

Remark 3.1 *Note that we are establishing that $\gamma_y(a) = x_b$ and $\gamma_y(b) = x_a$. This will make sense afterwards since we shall interpret γ_y as γ_x for reversed time.*

The set of restricted varied curves is defined by $\Gamma_\eta(\epsilon, t) := \tilde{\gamma}(t) + \epsilon\eta(t)$, where $\delta\tilde{\gamma} = \eta(t) = (\delta\gamma_x(t), \delta\gamma_x(t))$. In other words, we impose $\delta x = \delta y$.

Now, define a C^2 Lagrangian function $\mathcal{L} : \mathfrak{T}\mathbb{R}^d \to \mathbb{R}$ and the action functional $S : C^\infty(x_a, x_b; \mathbb{R}^d \times \mathbb{R}^d) \to \mathbb{R}$ given by

$$S(\tilde{\gamma}) := \int_a^b \mathcal{L}(\mathcal{V}_{\tilde{\gamma}})\, dt. \tag{11}$$

As proven in [5], Hamilton's principle with restricted curves as Γ_η and endpoint conditions (10), provides the so-called *restricted fractional Euler-Lagrange equations*: as *sufficient* conditions for $\tilde{\gamma}$ to be extremals of (11). If we particularize in the Lagrangian

$$\mathcal{L}(x, y, \dot{x}, \dot{y}, D_-^\alpha x, D_+^\alpha y) := L_x(x, \dot{x}) + L_y(y, \dot{y}) - [\![D_-^\alpha x, D_+^\alpha y]\!]_R, \tag{12}$$

where $L_x(x, \dot{x}) := \frac{1}{2}\dot{x}^T M \dot{x} - P(x)$, $L_y(y, \dot{y}) := \frac{1}{2}\dot{y}^T M \dot{y} - P(y)$ for $M = \mathrm{diag}(m_1, ..., m_d) \in \mathbb{M}^{d\times d}(\mathbb{R}^+)$ and $P : \mathbb{R}^d \to \mathbb{R}$ is a smooth function; and $[\![\cdot, \cdot]\!]_R$ is a symmetric bilinear form defined by $[\![D_-^\alpha x, D_+^\alpha y]\!]_R := (D_-^\alpha x)^T R\, D_+^\alpha y$, where $R = \mathrm{diag}(\rho_1, ..., \rho_d) \in \mathbb{M}^{d\times d}(\mathbb{R}^+)$; these equations are (see [5] for their general form):

$$M\,\ddot{x} + R\,D_-^{2\alpha} x + \nabla P(x) = 0, \;\Rightarrow\; M\,\ddot{x} + R\,\dot{x} + \nabla P(x) = 0, \tag{13a}$$

$$M\,\ddot{y} + R\,D_+^{2\alpha} y + \nabla P(y) = 0, \;\Rightarrow\; M\,\ddot{y} - R\,\dot{y} + \nabla P(y) = 0, \tag{13b}$$

when $\alpha = 1/2$, according to (7). We observe that equations (13) represent the dynamical evolution of a linearly damped mechanical system in both directions of time if we interpret that $\gamma_y(t)$ is $\gamma_x(t)$ in reversed time. Consequently, the system of second order differential equations (13) is invariant under time reversal, this is $t \to a + b - t$.

3.2 Fractional Legendre transform and restricted Hamiltonian dynamics

Let us define the *fractional Legendre transform* $\mathcal{FL} : \mathfrak{T}\mathbb{R}^d \to \mathfrak{T}^*\mathbb{R}^d$ as the fiber derivative for a Lagrangian function $\mathcal{L} : \mathfrak{T}\mathbb{R}^d \to \mathbb{R}$, i.e.

$$\begin{aligned} \mathcal{FL} : \mathfrak{T}_{\tilde\gamma}\mathbb{R}^d &\longrightarrow \mathfrak{T}^*_{\tilde\gamma}\mathbb{R}^d \\ \mathcal{V}_{\tilde\gamma} &\mapsto D_{\tilde\gamma}\mathcal{L}(\mathcal{V}_{\tilde\gamma}), \end{aligned} \tag{14}$$

where $D_{\tilde\gamma}$ denotes the partial derivative in the fiber $\mathcal{T}^{-1}(\tilde\gamma)$. Locally we have

$$\mathcal{FL}(\mathcal{V}_{\tilde\gamma}) = \left(\frac{\partial\mathcal{L}}{\partial\dot x}, \frac{\partial\mathcal{L}}{\partial\dot y}, \frac{\partial\mathcal{L}}{\partial D_-^\alpha x}, \frac{\partial\mathcal{L}}{\partial D_+^\alpha y}\right). \tag{15}$$

It is easy to check that $\mathcal{FL}$ is fiber preserving. Moreover, we will say that $\mathcal{FL}$ is *regular* if it is a diffeomorphism, and furthermore we will call $\mathcal{L}$ *regular* if that is the case. Under the hypothesis of regularity, we define the Hamiltonian function $\mathcal{H} : \mathfrak{T}^*\mathbb{R}^d \to \mathbb{R}$ by

$$\mathcal{H}(\mathcal{A}_{\tilde\gamma}) := \langle \mathcal{FL}(\mathcal{V}_{\tilde\gamma}), \mathcal{V}_{\tilde\gamma}\rangle - \mathcal{L}(\mathcal{V}_{\tilde\gamma}), \tag{16}$$

where the coordinates of $\mathcal{A}_{\tilde\gamma} := \mathcal{FL}(\mathcal{V}_{\tilde\gamma})$ are given in (9) and $\langle\cdot,\cdot\rangle : \mathfrak{T}_{\tilde\gamma}\mathbb{R}^d \times \mathfrak{T}^*_{\tilde\gamma}\mathbb{R}^d \to \mathbb{R}$ denotes the natural pairing.

Employing these elements, we can establish the following result.

Theorem 3.2 *A curve $\tilde\gamma : [a,b] \to \mathbb{R}^d \times \mathbb{R}^d$, subject to varied curves Γ_η, is an extremal of the action* (11) *if it satisfies the* restricted fractional Hamilton equations*:*

$$\dot x = \frac{\partial\mathcal{H}}{\partial p_x}, \quad D_-^\alpha x = \frac{\partial\mathcal{H}}{\partial p_x^\alpha}, \quad \dot p_x = D_-^\alpha p_y^\alpha - \frac{\partial\mathcal{H}}{\partial x}, \tag{17a}$$

$$\dot y = \frac{\partial\mathcal{H}}{\partial p_y}, \quad D_+^\alpha y = \frac{\partial\mathcal{H}}{\partial p_y^\alpha}, \quad \dot p_y = D_+^\alpha p_x^\alpha - \frac{\partial\mathcal{H}}{\partial y}, \tag{17b}$$

Proof. In the first place, we express the action (11) in terms of the Hamiltonian function, i.e.

$$S(\tilde\gamma) = \int_a^b \left\{p_x\dot x + p_y\dot y + p_x^\alpha D_-^\alpha x + p_y^\alpha D_+^\alpha y - \mathcal{H}(x,y,p_x,p_y,p_x^\alpha,p_y^\alpha)\right\} dt,$$

where we have employed (16) and the regularity of $\mathcal{F}L$. To find the extremals of S for restricted varied curves $\Gamma_\eta(\epsilon,t)$ we impose the usual critical condition, i.e. $\delta S := \frac{d}{d\epsilon}S(\Gamma_\eta)\big|_{\epsilon=0} = 0$. According to this, we have that

$$\begin{aligned} \delta S = \int_a^b \Big\{ &\delta p_x\left(\dot x - \frac{\partial\mathcal{H}}{\partial p_x}\right) + \delta p_y\left(\dot y - \frac{\partial\mathcal{H}}{\partial y}\right) + \delta p_x^\alpha\left(D_-^\alpha x - \frac{\partial\mathcal{H}}{\partial p_x^\alpha}\right) + \delta p_y^\alpha\left(D_+^\alpha y - \frac{\partial\mathcal{H}}{\partial p_y^\alpha}\right) \\ &+ \left(-\dot p_x + D_+^\alpha p_x^\alpha - \frac{\partial\mathcal{H}}{\partial x} - \dot p_y + D_-^\alpha p_y^\alpha - \frac{\partial\mathcal{H}}{\partial y}\right)\delta x\Big\} dt + p_x\delta x\Big|_a^b + p_y\delta y\Big|_a^b, \end{aligned}$$

where we have employed the constraints $\delta x = \delta y$, and have used integration by parts with respect to the total and fractional derivatives (6). From the endpoint conditions (10), all the boundary terms vanish, leading to

$$\begin{aligned}\delta S = \int_a^b \Big\{ &\delta p_x \left(\dot{x} - \frac{\partial \mathcal{H}}{\partial p_x}\right) + \delta p_x^\alpha \left(D_-^\alpha x - \frac{\partial \mathcal{H}}{\partial p_x^\alpha}\right) + \left(-\dot{p}_x + D_-^\alpha p_y^\alpha - \frac{\partial \mathcal{H}}{\partial x}\right) \delta x \\ &+ \delta p_y \left(\dot{y} - \frac{\partial \mathcal{H}}{\partial y}\right) + \delta p_y^\alpha \left(D_+^\alpha y - \frac{\partial \mathcal{H}}{\partial p_y^\alpha}\right) + \left(-\dot{p}_y + D_+^\alpha p_x^\alpha - \frac{\partial \mathcal{H}}{\partial y}\right) \delta x \Big\} dt.\end{aligned}$$

From this last expression of δS is easy to see that the restricted fractional Hamilton equations (17) are a sufficient condition for $\delta S = 0$; and the claim holds. □

As mentioned above, equations (17) are only sufficient conditions for the extremal curves.

Corollary 3.3 *If $\mathcal{L}$ is given by (12), with $L_x(x, \dot{x}) := \frac{1}{2}\dot{x}^T M \dot{x} - P(x)$, $L_y(y, \dot{y}) := \frac{1}{2}\dot{y}^T M \dot{y} - P(y)$ for M =diag$(m_1, ..., m_d) \in \mathbb{M}^{d\times d}(\mathbb{R}^+)$, R =diag$(\rho_1, ..., \rho_d) \in \mathbb{M}^{d\times d}(\mathbb{R}^+)$ and $P : \mathbb{R}^d \to \mathbb{R}$ is a smooth function; then the Hamiltonian function (16) reads*

$$\mathcal{H}(x, y, p_x, p_y, p_x^\alpha, p_y^\alpha) = H_x(x, p_x) + H_y(y, p_y) - [\![p_x^\alpha, p_y^\alpha]\!]_R^*, \tag{18}$$

where $H_x(x, p_x) := \frac{1}{2} p_x M^{-1} p_x^T + P(x)$, $H_y(y, p_y) := \frac{1}{2} p_y M^{-1} p_y^T + P(y)$ and $[\![\cdot, \cdot]\!]_R$ is a symmetric bilinear form defined by $[\![p_x^\alpha, p_y^\alpha]\!]_R^* := p_x^\alpha R^{-1} (p_y^\alpha)^T$.

Moreover, if $\alpha = 1/2$ and the Hamiltonian function is given by (18), then the restricted fractional Hamilton equations (17) are

$$\dot{x} = M^{-1} p_x^T, \quad D_-^{1/2} x = -R^{-1} (p_y^{1/2})^T, \quad \dot{p}_x = -p_x M^{-1} R - \nabla P(x)^T, \tag{19a}$$
$$\dot{y} = M^{-1} p_y^T, \quad D_+^{1/2} y = -R^{-1} (p_x^{1/2})^T, \quad \dot{p}_y = \quad p_y M^{-1} R - \nabla P(y)^T. \tag{19b}$$

Proof. From (12) and (15) we have that

$$p_x = \dot{x}^T M, \;\; p_y = \dot{y}^T M, \;\; p_x^\alpha = -(D_+^\alpha y)^T R, \;\; p_y^\alpha = -(D_-^\alpha x)^T R,$$

which is invertible. Then, (16) reads

$$\begin{aligned}\mathcal{H}(x, y, p_x, p_y, p_x^\alpha, p_y^\alpha) =& p_x M^{-1} p_x^T + p_y M^{-1} p_y^T - p_x^\alpha R^{-1} (p_y^\alpha)^T - p_y^\alpha R^{-1} (p_x^\alpha)^T \\ &- \frac{1}{2} p_x M^{-1} p_x^T + P(x) - \frac{1}{2} p_y M^{-1} p_y^T + P(y) + p_y^\alpha R^{-1} (p_x^\alpha)^T.\end{aligned}$$

From this expression it is straightforward to arrive to (18).

Regarding the second statement, for (18) the restricted fractional Hamilton equations read

$$\dot{x} = M^{-1} p_x^T, \quad D_-^\alpha x = -R^{-1} (p_y^\alpha)^T, \quad \dot{p}_x = -D_-^\alpha (D_-^\alpha x)^T R - \nabla P(x)^T, \tag{20a}$$
$$\dot{y} = M^{-1} p_y^T, \quad D_+^\alpha y = -R^{-1} (p_x^\alpha)^T, \quad \dot{p}_y = -D_+^\alpha (D_+^\alpha y)^T R - \nabla P(y)^T. \tag{20b}$$

When we particularize in $\alpha = 1/2$, (7) applies; furthermore, inserting the dynamical equation $\dot{x} = M^{-1} p_x^T$ (resp. $\dot{y} = M^{-1} p_y^T$) into the dynamical equation of $\dot{p}_x$ (resp. $\dot{p}_y$) in the last expression, we obtain directly (19). □

Remark 3.4 *For each coordinate, the dynamical equations* (19) *are*

$$\dot{x}^i = \frac{1}{m_i}(p_x)_i, \quad (\dot{p}_x)_i = -\frac{\rho_i}{m_i}(p_x)_i - \partial_i P(x), \tag{21a}$$

$$\dot{y}^i = \frac{1}{m_i}(p_y)_i, \quad (\dot{p}_y)_i = \quad \frac{\rho_i}{m_i}(p_y)_i - \partial_i P(y), \tag{21b}$$

for $i = 1, ..., d$. We observe that these are the Hamiltonian dynamics of linearly damped mechanical systems. Defining the energy as $E_i(\xi^i, (p_\xi)_i) = (p_\xi)_i^2/2m_i + P(\xi^i)$, we observe that the $x-$system is dissipative, whereas the $y-$system gains energy at the same rate. This is consistent with the interpretation of the y-system as the x-system in reversed time as discussed after (13) *[5]. Accordingly, in the sequel we shall consider $t = b$ as the final time for the x-system and initial time for the y-system (conversely for $t = a$).*

3.3 Controlled restricted Hamiltonian dynamics

We are only going to consider external forces coming from actuators steering the system (since the damping is obtained through the fractional approach). We double our control space, accordingly with the phase space. Thus, we consider $U \equiv \mathbb{R}^{m_u}$, $V \equiv \mathbb{R}^{m_v}$ the control spaces, with $m_u, m_v \leq d$, and express the external controlled forces by

$$\begin{aligned} f_x : U &\to T_x^*\mathbb{R}^d, & f_y : V &\to T_x^*\mathbb{R}^d, \\ u &\mapsto f_x(u), & v &\mapsto f_y(v), \end{aligned} \tag{22}$$

We pick $T_x^*\mathbb{R}^d$ for both forces because we are going to consider restricted varied curves Γ_η. Moreover, we are allowed to consider the sum $f_x(u) + f_y(v)$ since $T_x^*\mathbb{R}^d$ is a vector space. The next theorem stands for the obtaining of the controlled restricted fractional Hamilton equations (the proof is equivalent to Theorem 3.2's.).

Theorem 3.5 *A curve $\tilde{\gamma} : [a, b] \to \mathbb{R}^d \times \mathbb{R}^d$, subject to restricted varied curves Γ_η satisfies the Lagrange-d'Alembert principle (fractional Hamiltonian version with controlled external forces)*

$$\delta \int_a^b \left\{ p_x\dot{x} + p_y\dot{y} + p_x^\alpha D_-^\alpha x + p_y^\alpha D_+^\alpha y - \mathcal{H} \right\} dt + \int_a^b (f_x(u) + f_y(v))\, \delta x \, dt = 0,$$

if it satisfies the controlled restricted fractional Hamilton equations*:*

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x}, \quad D_-^\alpha x = \frac{\partial \mathcal{H}}{\partial p_x^\alpha}, \quad \dot{p}_x = D_-^\alpha p_y^\alpha - \frac{\partial \mathcal{H}}{\partial x} + f_x(u), \tag{23a}$$

$$\dot{y} = \frac{\partial \mathcal{H}}{\partial p_y}, \quad D_+^\alpha y = \frac{\partial \mathcal{H}}{\partial p_y^\alpha}, \quad \dot{p}_y = D_+^\alpha p_x^\alpha - \frac{\partial \mathcal{H}}{\partial y} + f_y(v). \tag{23b}$$

4 FRACTIONAL OPTIMAL CONTROL PROBLEM

In this section, we extend the OCP 2.1, to the fractional scenario. This is, we are going to consider controlled restricted fractional Hamilton dynamics (23). Furthermore, we will be mainly interested in Hamiltonian functions with the form (18) and

$$H_x(x,p_x)=K(p_x)+P(x),\qquad H_y(y,p_y)=K(p_y)+P(y).$$

In such a case, the fractional momenta p_x^α, p_y^α are directly related to the coordinates x, y through the second equations in (20a) and (20b). Thus, they are no longer independent variables and we can remove them from our phase space (this is, instead of $\mathcal{T}^*\mathbb{R}^d$ we shall consider $T_x^*\mathbb{R}^d\times T_y^*\mathbb{R}^d$). According to this, we pick the cost function $\mathcal{C}:T_x^*\mathbb{R}^d\times T_y^*\mathbb{R}^d\times U\times V\rightarrow\mathbb{R}$ and Mayer term $\tilde\Phi:T_x^*\mathbb{R}^d\times T_y^*\mathbb{R}^d\rightarrow\mathbb{R}$. We define the Fractional Optimal Control Problem (FOCP), in its Hamiltonian version, as follows.

Problem 4.1 (Fractional Optimal Control Problem)

$$\begin{aligned}
\min_{(x,p_x,y,p_y,u,v)}\mathcal{J}(x,p_x,y,p_y,u,v)=&\int_a^b\mathcal{C}(x,p_x,y,p_y,u,v)dt && (24a)\\
&\qquad+\tilde\Phi(x(b),p_x(b),y(a),p_y(a)),\\
\text{subject to}\qquad \dot x=&\ \partial\mathcal{H}/\partial p_x, && (24b)\\
\dot p_x=&-(D_-^{2\alpha}x)^TR-\partial\mathcal{H}/\partial x+f_x(u), && (24c)\\
\dot y=&\ \partial\mathcal{H}/\partial p_y, && (24d)\\
\dot p_y=&-(D_+^{2\alpha}y)^TR-\partial\mathcal{H}/\partial y+f_y(v), && (24e)\\
(x(a),p_x(a),y(b),p_y(b))=&\ (x^a,p_x^a,y^b,p_y^b). && (24f)
\end{aligned}$$

We establish necessary optimality conditions for feasible curves $\tilde\eta(\cdot)=(x(\cdot),p_x(\cdot),y(\cdot),p_y(\cdot),u(\cdot),v(\cdot))$ (which are those curves satisfying (24b)-(24f); they will be optimal if moreover they satisfy (24a)) by means of the Euler-Lagrange equations of the augmented cost functional

$$\begin{aligned}\tilde{\mathcal{S}}(\tilde\eta,\tilde\lambda)=\int_a^b\Big\{&\mathcal{C}(x,p_x,y,p_y,u,v)+\langle\lambda^x,\dot x-\partial\mathcal{H}/\partial p_x\rangle+\langle\lambda^{p_x},\dot p_x+(D_-^{2\alpha}x)^TR+\partial\mathcal{H}/\partial x-f_x(u)\rangle\\
&+\langle\lambda^y,\dot y-\partial\mathcal{H}/\partial p_y\rangle+\langle\lambda^{p_y},\dot p_y+(D_+^{2\alpha}y)^TR+\partial\mathcal{H}/\partial y-f_y(v)\rangle\Big\}dt+\tilde\Phi(x(b),p_x(b),y(a),p_y(a)),\end{aligned}\tag{25}$$

where now $\tilde\lambda(\cdot)=(\lambda^x(\cdot),\lambda^{p_x}(\cdot),\lambda^y(\cdot),\lambda^{p_y}(\cdot))$ These Euler-Lagrange equations are

$$\begin{aligned}
\dot x=&\ \frac{\partial K}{\partial p_x}, & \dot y=&\ \frac{\partial K}{\partial p_y}, && (26a)\\
\dot p_x=&-(D_-^{2\alpha}x)^TR-\frac{\partial P}{\partial x}+f_x(u), & \dot p_y=&-(D_+^{2\alpha}y)^TR-\frac{\partial P}{\partial y}+f_y(v), && (26b)\\
\dot\lambda^x=&\ (D_+^{2\alpha}\lambda^{p_x})R+\lambda^{p_x}\frac{\partial^2P}{\partial x^2}+\frac{\partial\mathcal{C}}{\partial x}, & \dot\lambda^y=&\ (D_-^{2\alpha}\lambda^{p_y})R+\lambda^{p_y}\frac{\partial^2P}{\partial y^2}+\frac{\partial\mathcal{C}}{\partial y}, && (26c)
\end{aligned}$$

$$\dot{\lambda}^{p_x} = -\lambda^x \frac{\partial^2 K}{\partial p_x^2} + \frac{\partial \mathcal{C}}{\partial p_x}, \qquad \dot{\lambda}^{p_y} = -\lambda^y \frac{\partial^2 K}{\partial p_y^2} + \frac{\partial \mathcal{C}}{\partial p_y}, \tag{26d}$$

$$0 = \frac{\partial \mathcal{C}}{\partial u} - \lambda^{p_x} \frac{\partial f_x}{\partial u}, \qquad 0 = \frac{\partial \mathcal{C}}{\partial v} - \lambda^{p_y} \frac{\partial f_y}{\partial v}, \tag{26e}$$

with endpoint conditions

$$\begin{aligned} x(a) = x^a, \quad p_x(a) = p_x^a, \quad \lambda^x(b) = -\frac{\partial \tilde{\Phi}}{\partial x}\Big|_b, \quad \lambda^{p_x}(b) = -\frac{\partial \tilde{\Phi}}{\partial p_x}\Big|_b, \\ y(b) = y^b, \quad p_y(b) = p_y^b, \quad \lambda^y(a) = -\frac{\partial \tilde{\Phi}}{\partial y}\Big|_a, \quad \lambda^{p_y}(a) = -\frac{\partial \tilde{\Phi}}{\partial p_y}\Big|_a. \end{aligned} \tag{27}$$

It is important to point out that in the variational process leading to (26), (27) we have considered $\delta x(a) = \delta p_x(a) = \delta y(b) = \delta p_y(b) = 0$, while $\delta x(b), \delta p_x(b), \delta y(a), \delta p_y(a)$ free. We observe that they conform a system of algebraic-differential equations with fractional terms and mixed (initial-final) endpoint conditions. From the algebraic equations (26e), according to the implicit function theorem we observe that if the matrix

$$\begin{bmatrix} \frac{\partial^2 \mathcal{C}}{\partial u^2} - \lambda^{p_x} \frac{\partial^2 f_x}{\partial u^2} & \frac{\partial^2 \mathcal{C}}{\partial v \partial u} \\ \frac{\partial^2 \mathcal{C}}{\partial u \partial v} & \frac{\partial^2 \mathcal{C}}{\partial v^2} - \lambda^{p_y} \frac{\partial^2 f_y}{\partial v^2} \end{bmatrix}$$

is regular, then we can determine u, v as a functions of the rest of variables, i.e. $u = u(x, p_x, y, p_y, \lambda^{p_x}, \lambda^{p_y})$, $v = v(x, p_x, y, p_y, \lambda^{p_x}, \lambda^{p_y})$. Inserting those into (26b)-(26d) we end up with a pure system of differential equations with fractional terms and mixed endpoint conditions where the control variables are absent. However, we observe that still in that case the x and y sides are coupled through the terms involving f_x, f_y and $\mathcal{C}$. Obviously, the decoupling of both sides depends on the particular form of these functions.

4.1 Example

We consider the expample of a linearly damped harmonic oscillator (both in the x and y sides) and minimum effort with external forces which are linear in the control variables. Let us consider $d = m_u = m_v = 1$. We pick $\alpha = 1/2$, $R = (\rho_1) = 1$ and $f_x(u) = u$, $f_y(v) = v$, the cost function $\mathcal{C}(x, p_x, y, p_y, u, v) = u^2/2 + v^2/2$, and Hamiltonian functions $H_x(x, p_x) = p_x^2/2 + x^2/2$, $H_y(y, p_y) = p_y^2/2 + y^2/2$. In this case, the necessary optimality conditions (26) read

$$\begin{aligned} &\dot{x} = p_x, && \dot{p}_x = -p_x - x + u; && \dot{y} = p_y, && \dot{p}_y = p_y - y + v; \\ &\dot{\lambda}^x = \lambda^x + \lambda^{p_x}, && \dot{\lambda}^{p_x} = -\lambda^x; && \dot{\lambda}^y = -\lambda^y + \lambda^{p_y}, && \dot{\lambda}^{p_y} = -\lambda^y; \\ &0 = u - \lambda^{p_x}; && && 0 = v - \lambda^{p_y}. \end{aligned}$$

We have taken into account that $D_-^1 x = \dot{x}$, $D_+^1 y = -\dot{y}$ (according to (7)) in eqs. (26b) and $\dot{x} = p_x$, $\dot{y} = p_y$ in eqs. (26b). Moreover, in (26c) we have that $D_+^1 \lambda^{p_x} = -\dot{\lambda}^{p_x}$, $D_-^1 \lambda^{p_y} = \dot{\lambda}^{p_y}$ and $\dot{\lambda}^{p_x} = -\lambda^x$, $\dot{\lambda}^{p_y} = -\lambda^y$ from (26d). Taking into account that $u = \lambda^{p_x}$, $v = \lambda^{p_y}$, we end up with a linear system of ordinary differential equations, where the x

and y sides are decoupled, circumstance which is strongly dependent on the particular choice $\mathcal{C} = u^2/2 + v^2/2$.

Let us now focus on the x-system. It is interesting to observe that the dynamics of the costates, say

$$\begin{bmatrix} \dot{\lambda}^x \\ \dot{\lambda}^{p_x} \end{bmatrix} = \Lambda_1 \begin{bmatrix} \lambda^x \\ \lambda^{p_x} \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}, \tag{28}$$

is *not* the same that one obtains from (4) choosing $H(x, p_x) = p_x^2/2 + x^2/2$, $f_H(x, p_x, u) = -p_x + u$ and $C = u^2/2$, as one would expect. The costate dynamics following from these choices is

$$\begin{bmatrix} \dot{\lambda}^x \\ \dot{\lambda}^{p_x} \end{bmatrix} = \Lambda_2 \begin{bmatrix} \lambda^x \\ \lambda^{p_x} \end{bmatrix}, \quad \Lambda_2 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}. \tag{29}$$

However, (28) is the costate dynamics that one obtains setting the forced dynamics as

$$\dot{x} = p_x, \qquad \dot{p}_x = -\dot{x} - x + u, \tag{30}$$

instead of the natural choice when we work on T^*Q, i.e.

$$\dot{x} = p_x, \qquad \dot{p}_x = -p_x - x + u, \tag{31}$$

when we define the augmented Lagrangian (3). However, if we fix u as an external parameter, it is straightforward to check that (30) and (31) determine the same regular submanifold of TT^*Q, which has coordinates $(x, p_x, \dot{x}, \dot{p}_x)$, and therefore one expects to find certain transformation linking (28) and (29), or in other words linking Λ_1 and Λ_2, which accounts for a linear transformation $\begin{bmatrix} \lambda^x \\ \lambda^{p_x} \end{bmatrix} = T \begin{bmatrix} \tilde{\lambda}^x \\ \tilde{\lambda}^{p_x} \end{bmatrix}$. Indeed, one can check that

$$T^{-1}\Lambda_1 T = \Lambda_2 \ \text{ for } \ T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ such that}$$
$$dc - ad - ab = 0, \ \ d^2 + b^2 - c^2 - a^2 + ac - db = 0, \ \ ad - cb \neq 0.$$

The conditions above are satisfied if $a = b = d = 1$ and $c = 0$. In such a case, one can check that

$$\langle \lambda^x, \dot{x} - p_x \rangle + \langle \lambda^{p_x}, \dot{p}_x + p_x + x + u \rangle = \langle \tilde{\lambda}^x, \dot{x} - p_x \rangle + \langle \tilde{\lambda}^{p_x}, \dot{p}_x + \dot{x} + x + u \rangle$$

in the definition of (3). T can be alternatively understood as a linear transformation in the space of costates λ or in the submanifold defined by dynamics (30) and (31).

5 CONCLUSIONS

We establish the Fractional Optimal Control Problem for controlled Hamiltonian dynamics and obtain the necessary optimality conditions. These determine the dynamical evolution of coordinates, momenta, costates and control variables, and we observe that they are a system of algebraic-differential equations with fractional terms and mixed (initial-final) endpoint conditions. Under some regularity conditions, the control variables can be solved in terms of the rest, leading to a system of pure fractional differential equations as optimality conditions. In the treated example (linearly damped harmonic oscillator and minimum effort with external forces which are linear in the control variables), the deviation from the expected behaviour of the costates, with respect to the non-fractional case, is explained in terms of the geometry of higher cotangent bundles (concretely TT^*Q).

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