Operator splitting approach for coupling Stokes flow and nonlinear systems of ODEs

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ABSTRACT

Multiscale coupling of nonlinear lumped and distributes fluid flow models is often necessary when modeling complex biological systems such as blood flow through the cardiovascular system. We develop a novel technique based on operator splitting for the time discretization of coupled systems of Stokes equations and ordinary differential equations (ODEs) that allows to solve separately and sequentially the Stokes problem and the ODEs without the need of sub-iterations [1].

Consider a non-stationary Stokes flow in a rigid domain $\Omega \subset \mathbb{R}^d$, with $d = 2, 3$, coupled with a zero-dimensional (0D) network via a RC buffer (Fig.1) [2]. To ensure the well-posedness of the problem, at the coupling interface Σ, we impose continuity of pressure, via a natural Neumann condition, and continuity of flow. Let $\underline{\boldsymbol{v}}$ and p be the fluid velocity and pressure in Ω and $\underline{\boldsymbol{\mathcal{P}}}$ the vector of pressures at the nodes of the RC buffer and of the 0D network.

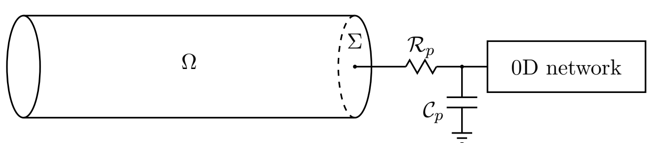

Fig.1 Sketch of a coupled problem between a fluid flow in Ω and a zero-dimensional (0D) network. The coupling occurs on Σ via a RC buffer of resistance $\mathcal{R}_p$ and capacitance $\mathcal{C}_p$.

The splitting scheme proposed, based on the first-order operator splitting scheme [3], consists of two steps that communicate via the initial conditions as follows: given Δt and let $t^n = n\Delta t$, for any $n \geq 0$

Step 1 given $\underline{\boldsymbol{v}}^n$ and $\underline{\boldsymbol{\mathcal{P}}}^n$ solve the Stokes equations in Ω coupled with the RC buffer in (t^n, t^{n+1}) and set $\underline{\boldsymbol{v}}^{n+\frac{1}{2}} = \underline{\boldsymbol{v}}(t^{n+1})$, $\underline{\boldsymbol{\mathcal{P}}}^{n+\frac{1}{2}} = \underline{\boldsymbol{\mathcal{P}}}(t^{n+1})$, and $p^{n+1} = p(t^{n+1})$;

Step 2 given $\underline{\boldsymbol{v}}^{n+\frac{1}{2}}$ and $\underline{\boldsymbol{\mathcal{P}}}^{n+\frac{1}{2}}$ solve the 0D network system of ODEs in (t^n, t^{n+1}) and set $\underline{\boldsymbol{v}}^{n+1} = \underline{\boldsymbol{v}}(t^{n+1})$, $\underline{\boldsymbol{\mathcal{P}}}^{n+1} = \underline{\boldsymbol{\mathcal{P}}}(t^{n+1})$.

The energy of the semi-discrete problem mirrors the behavior of the energy of the full coupled system, thereby providing unconditional stability to the proposed splitting method. The scheme presented can be extended to a second-order scheme in time, to Navier-Stokes equations and fluid-structure interactions in Ω by combining the proposed scheme with other operator splitting techniques already developed [3, 4].

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