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SMOOTHING OF YIELD SURFACES AND A REFORMULATION OF MULTI-SURFACE PLASTICITY

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Abstract. In this work we describe a procedure for the smoothing of non-regular yield surfaces and plastic potential functions. We also present several application examples corresponding to different well-known cases. Moreover, we show that a multi-surface plasticity model can be reduced to a model with a single yield surface by using the same smoothing procedure.

1 INTRODUCTION

Yield Surfaces and Plastic Potential functions are two essential ingredients in Plasticity Theory; the former are defined as surfaces in the stress space that bound the elastic domain; the latter fix the direction of the vector of incremental plastic strains that appears under plastic loading. Many equations have been proposed to fit the shape of different yield surfaces and plastic potential functions and improving the performance of material modeling. Some of these equations produce geometrical singularities that imply the appearance of different conceptual and numerical problems due basically to that for certain loading states the vector of incremental plastic strains cannot be properly defined as the product of a plastic multiplier times the gradient of a yield function or a plastic potential function.

A celebrated theoretical solution due to Koiter [1] consists in writing the plastic strain rate vector at a singular point as a linear combination of the gradients of the concurrent plastic potential functions; the coefficients of the linear combination can be determined by solving a linear system of equations which is obtained by imposing the consistency conditions associated to each one of the involved yield surfaces. Further developments can be found, for instance, in [2,3,4] and the references therein.

Regarding the numerical implementation of models with singularities, often *ad hoc* corner rounding techniques are employed, though the programming may be laborious and the problem of the singularities may remain if higher-order derivatives of the involved functions

are needed. In some particular cases closed-form expressions for smooth yield surfaces and plastic potential functions have been derived. Occasionally, the smooth approximations fit the experimental data better than the original singular model; this is the case, for instance, of Mohr-Coulomb's (MC) surface, for which some smooth variations are available in the literature [5,6,7].

Here we describe a smoothing technique based on the algebraic composition of several implicit equations corresponding to different yield surfaces or plastic potential functions in order to produce a single implicit equation corresponding to a family of regular surfaces. The presented procedures have a wide range of application and their versatility allow us to tackle different variants of the smoothing problem. For instance, the non-regular points may appear due to the symmetries of the stress tensor, as it is the case of the MC model. In other cases, the singularities appear because the elastic domain is defined by means of different surface equations that correspond to different plastic mechanisms; this is the case of the so-called multi-surface models. With the proposed method, the reduction of a given multi-surface model to a model with a single regular yield surface is straightforward, which can be viewed as a reformulation of multi-surface plasticity.

In the following Sections we describe and illustrate the application of the presented approach to the smoothing of different yield surfaces; nevertheless, the presented procedures are applicable to yield surfaces and to plastic potential functions indistinctly, and they can be used both in the frameworks of associated and non-associated plasticity.

2 BRIEF DESCRIPTION OF THE SMOOTHING TECHNIQUE

Let us start with the simple case of the approximation of the boundary of a square by a smooth curve. We can draw the family of curves defined by the implicit equations $x^{2n} + y^{2n} = 1$ and study its behavior. In Figure 1 we can observe that, as n increases, the corresponding curve, which is a circumference in the beginning, becomes a square. Nevertheless, the obtained curves are perfectly regular in all cases.

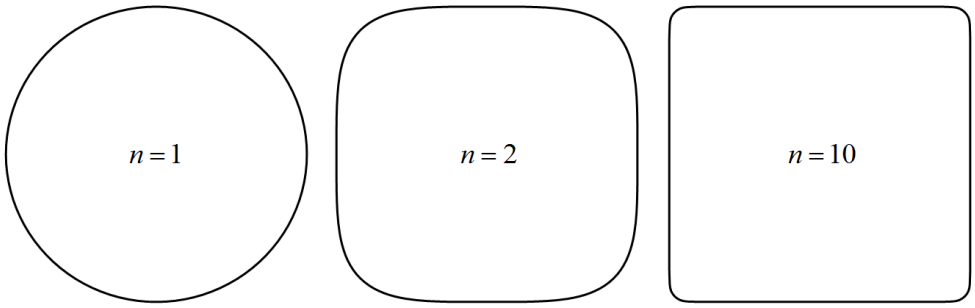

Figure 1: Evolution from a circumference to a square.

Let us note that the interior of the square can be described with the set $\{(x, y) \in \mathbb{R}^2 \,|\, F_1(x, y) < 1,\ F_2(x, y) < 1\}$, where functions $F_1(x, y) = x^2$ and $F_2(x, y) = y^2$ are non-negative.

2.1 Foundations of the method

If Ω is a region of $\mathbb{R}^m$ which can be described in the form

$$\Omega = \left\{ \mathbf{x} \in \mathbb{R}^m \,\middle|\, F_i(\mathbf{x}) < 1,\ i = 1,\ldots,k \right\}$$

for some k non-negative functions $F_i : \mathbb{R}^m \to \mathbb{R}^+$, $\mathbb{R}^+ = [0,\infty)$, then the sequence of sets $\{A_n\}_{n=1}^{\infty}$ defined by

$$A_n = \left\{ \mathbf{x} \in \mathbb{R}^m \,\middle|\, \sum_{i=1}^{k} \left(F_i(\mathbf{x}) \right)^n < 1 \right\}$$

grows up to Ω -*i.e.*, in terms of the Theory of Sets, $A_n \uparrow \Omega$-.

This result can be easily proved: if $\mathbf{x} \in A_n$, then $\left(F_i(\mathbf{x})\right)^{n+1} \le \left(F_i(\mathbf{x})\right)^n$, and, therefore, $\mathbf{x} \in A_{(n+1)}$. Moreover, if $\mathbf{x} \in \Omega$, then some $n_x \in \mathbf{N}$ there exists such that $\mathbf{x} \in A_{n_x}$; to see this, it is enough to see that $\sum_{i=1}^{k} \left(F_i(\mathbf{x})\right)^n \le k \left(\max\{F_i(\mathbf{x})\}\right)^n$; thus, one can choose $n_x > \dfrac{\ln(1/k)}{\ln \max\{F_i(\mathbf{x})\}}$ if $\max\{F_i(\mathbf{x})\} > 0$ and $n_x = 1$ if $\max\{F_i(\mathbf{x})\} = 0$.

As a consequence of that, if F_i are regular functions, then the surface implicitly defined as $F(\mathbf{x}) = \sum_{i=1}^{k} \left(F_i(\mathbf{x})\right)^n = 1$ -that is, ∂A_n- is also regular and can be used as a smooth approximation of $\partial\Omega$.

The above reasoning can be easily adapted to a more general expression like $F(\mathbf{x}) = \sum_{i=1}^{k} \left(F_i(\mathbf{x})\right)^{\alpha_i} = 1$, where the exponents α_i are different and not necessarily integers. Moreover, if Ω is described by $\Omega = \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \widetilde{F}_i(\mathbf{x}) < a_i,\ i = 1,\ldots,k \right\}$, where $\widetilde{F}_i$ are not positive functions, it is always possible to obtain an equivalent description of Ω in the form $\Omega = \left\{ \mathbf{x} \in \mathbb{R}^m \,\middle|\, F_i(\mathbf{x}) < 1,\ i = 1,\ldots,k \right\}$, where F_i are positive functions, just by taking, for example, $F_i(\mathbf{x}) = e^{\widetilde{F}_i(\mathbf{x}) - a_i}$.

This smoothing procedure is well-known, especially in the field of Computer Graphics. As a reference, the work [8] can be mentioned. A more detailed description of the technique can be found in [9], where it is applied to obtain numerical solutions for non-linear optimization problems on non-regular domains.

2.2 A general application example

Let us consider the functions

$$F_1(x,y,z) = x^2 + y^2 + z^2 - 1 \qquad\qquad F_2(x,y,z) = (x - 1.1)^2 + y^2 + z^2 - 0.35^2$$

$$F_3(x,y,z)=\frac{1}{2}\left(\frac{x-z}{1.2}\right)^2+\left(\frac{y-1.3}{0.6}\right)^2+\frac{1}{2}\left(\frac{x+z}{0.6}\right)^2-1$$

$$F_4(x,y,z)=\left(\frac{\sqrt{(y+0.6)^2+(z-0.6)^2}-1}{0.3}\right)^2+\left(\frac{x}{0.3}\right)^2-1,$$

and the domain $\Omega=\left\{x\in\mathbb{R}^3:\min\{F_1,F_2,F_3,F_4\}>0\right\}$. Equations $F_1=F_2=F_3=F_4=0$ describe two spheres, an ellipsoid and a torus, respectively, and Ω is the intersection of the exterior of these surfaces. Figure 2 displays $\partial\Omega$, which is a piecewise regular surface.

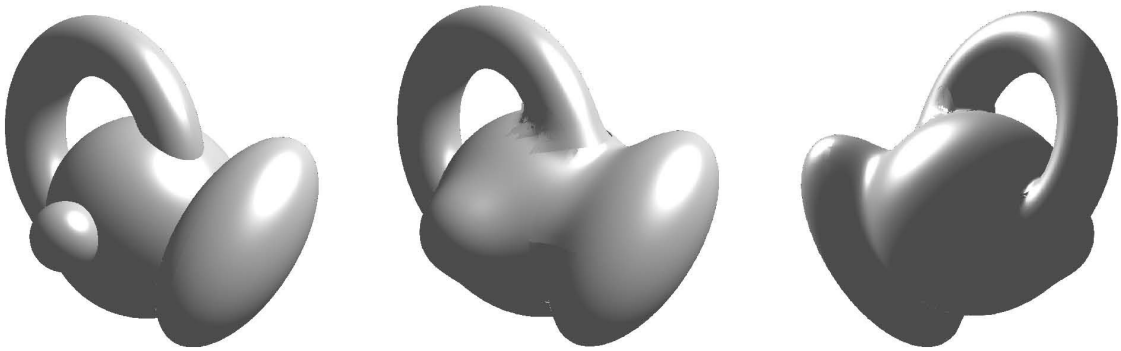

Figure 2: The boundary of Ω (left) and two perspectives of a smooth approximating surface (center and right).

We have $\Omega=\left\{x\in\mathbb{R}^3:\min\left\{e^{F_1},e^{F_2},e^{F_3},e^{F_4}\right\}>1\right\}=\left\{x\in\mathbb{R}^3:\max\left\{e^{-F_1},e^{-F_2},e^{-F_3},e^{-F_4}\right\}<1\right\}$, which allows us to define the implicit equations of smooth surfaces that tend to $\partial\Omega$ as, for instance, $S_p=\left\{x\in\mathbb{R}^3:F(x,y,z,p)=e^{-pF_1}+e^{-pF_2}+e^{-pF_3}+e^{-pF_4}=1\right\}$. Figure 2 shows two perspectives of the surface S_3.

2.3 The derivatives of the approximating smooth functions

With regards to the derivatives of *F,* note that $\frac{\partial F}{\partial x_i}=\sum_{j=1}^{k}\alpha_j\left(F_j(\mathbf{x})\right)^{\alpha_j-1}\frac{\partial F_j}{\partial x_i}$, and, therefore, the derivative $\frac{\partial F}{\partial x_i}$ is simply a linear combination of the derivatives $\frac{\partial F_j}{\partial x_i}$. The coefficients or weights $w_i=\alpha_i\left(F_i(\mathbf{x})\right)^{\alpha_i-1}$ of the linear combination describe the 'proximity' of $\mathbf{x}$ to each one of the different hyper-surfaces implicitly defined by $F_i(\mathbf{x})=1$. Let us finish by observing that if $\Omega=\left\{\mathbf{x}\in\mathbb{R}^m\,\middle|\,\widetilde{F}_i(\mathbf{x})<a_i,\ i=1,\ldots,k\right\}$, then the surface $\widetilde{F}(\mathbf{x})=\sum_{i=1}^{k}e^{\alpha\left(\widetilde{F}_i(\mathbf{x})-a_i\right)}=1$, where we have supposed that all the exponents α_i are equal, is equivalent to the surface

$F(\mathbf{x}) = \ln\left(\sum_{i=1}^{k} e^{\alpha\left(\widetilde{F}_i(\mathbf{x})-a_i\right)}\right)^{\frac{1}{\alpha}} = 0$, and that this yield function satisfies $\frac{\partial F}{\partial x_i} = \sum_{j=1}^{n} w_j \frac{\partial F_j}{\partial x_i}$, where the weights $w_i = \frac{e^{\alpha\left(\widetilde{F}_i(\mathbf{x})-a_i\right)}}{\sum_{j=1}^{n} e^{\alpha\left(\widetilde{F}_j(\mathbf{x})-a_j\right)}}$ take values between 0 and 1.

3 PRIMARY SMOOTHING

If the material is isotropic, then the yield functions are commonly written in terms of invariants of the stress tensor. We will work here with three classical invariants: $p = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$, the average stress, $J = \sqrt{\frac{1}{6}\left((\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2\right)}$, a deviatoric stress, and $\theta = \arctan\left(-\frac{1}{\sqrt{3}}\frac{\sigma_1 - 2\sigma_2 + \sigma_3}{\sigma_1 - \sigma_3}\right)$, Lode's angle. We can suppose now that an expression for a yield surface of the model is $F(p, J, \theta, \chi) = 1$, where $F(p, J, \theta, \chi) \geq 0$ and F is a regular function. This yield surface defines an elastic domain in the first sextant $S = \left\{(\sigma_1, \sigma_2, \sigma_3) \in \mathbb{R}^3 \,\middle|\, \sigma_1 \geq \sigma_2 \geq \sigma_3\right\}$. By using symmetries, we can obtain the corresponding elastic domain Ω in the stress space.

In this context, we distinguish two categories of smoothing: the first one, that we will call *primary smoothing*, corresponds to the fact that each one of the yield functions involved in the definition of a given model could need to be smoothed because of the apparition of corners after the application of the symmetries; if it would occur, this primary smoothing should be made even with a single-surface model. On the other hand, if the model requires different yield surfaces, a *multi-surface smoothing* could be needed to have a regular transition between all of them. In this Section we focus on the case of primary smoothing; multi-surface smoothing will be dealt with in Section 4.

3.1 Drucker-Prager's (DP) surface

Let us consider a yield function of the form $\widetilde{F}(p, J, a, b) = -ap + J - b$ for some $a > 0$, $b \geq 0$. The function is independent of the third invariant θ, and the surface $\widetilde{F}(p, J, a, b) = 0$ is a half cone in the stress space.

The surface $-ap + J - b = 0$ can be represented in the two-dimensional auxiliary space $p - J$, and also the surface $\widetilde{F}(p, -J, a, b) = -ap - J - b = 0$. Then, a smooth surface that approximates the cone is $F(p, J, a, b) = e^{\alpha(-ap+J-b)} + e^{-\alpha(ap+J+b)} = 2e^{-\alpha(ap+b)} ch\alpha J = 1$. Figure 3 shows the shape of the approximation in a generic case. The relationship between the power

α and the maximum absolute error ε is $\alpha = \frac{\ln 2}{a\varepsilon}$, and an alternative expression for the equation of the smoothed half cone is $2^{-\left(\frac{ap+b}{a\varepsilon}-1\right)} \cosh \frac{J \ln 2}{a\varepsilon} = 1$.

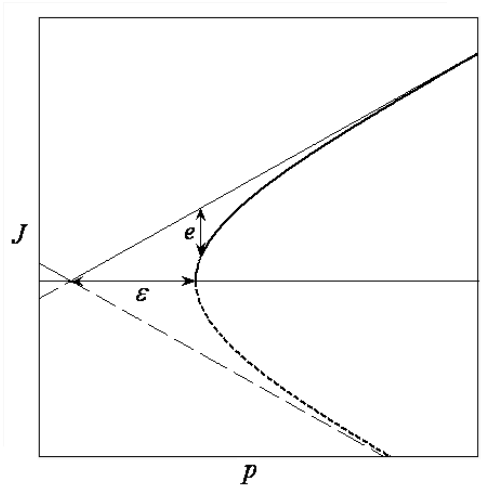

Figure 3: a smooth approximation of a DP surface.

On the other hand, it is frequent to describe the yield surfaces by means of dimensionless implicit equations. In that case, we can consider for instance the expression $\widetilde{F}(p,J,a,b) = \frac{J}{ap+b} = 1$. A smooth approximation of the half cone can be obtained by combining the yield function $\left|\frac{J}{ap+b}\right|$ with the plane $p = -\frac{b}{a} + \varepsilon$, $\varepsilon > 0$, in the form $\left|\frac{J}{ap+b}\right|^{\alpha} + e^{-\beta\left(\frac{ap+b}{a\varepsilon}-1\right)} = 1$, where $\alpha > 1$ -this restriction is necessary for guarantying the differentiability of the yield surface in the hydrostatic axis $J = 0$- and $\beta > 0$. As can be seen, in this case we have two independent parameters for controlling the shape of the approximation. It is necessary to choose carefully the values of these parameters in order to guaranty that the shape of the smooth surface is acceptable; in particular, we recommend to use $\beta \geq 1$.

3.2 Mohr-Coulomb's (MC) surface

In this case we can choose for the yield surface the dimensionless expression $\widetilde{F}(p,J,\theta,c',\varphi') = g(\theta) - \frac{p+a}{J} = 0$, where $g(\theta) = \frac{\cos\theta}{\sin\varphi'} + \frac{1}{\sqrt{3}}\sin\theta$, $a = c'\cot\varphi'$ and c', φ' are the cohesion and the internal friction angle of the material, respectively. The above yield surface is a straight line in the π-plane; if we apply the symmetries with respect to the axis of principal strains, we obtain the following six functions

$$F_1(p,J,\theta,c',\varphi') = \widetilde{F}(p,J,\theta,c',\varphi'), F_2(p,J,\theta,c',\varphi') = F_1\left(p,J,\frac{\pi}{3}-\theta,c',\varphi'\right)$$

$$F_3(p,J,\theta,c',\varphi') = F_1\left(p,J,-\frac{\pi}{3}-\theta,c',\varphi'\right), F_4(p,J,\theta,c',\varphi') = F_1(p,J,\pi-\theta,c',\varphi'),$$

$$F_5(p,J,\theta,c',\varphi') = F_1\left(p,J,\frac{2\pi}{3}+\theta,c',\varphi'\right),\ F_6(p,J,\theta,c',\varphi') = F_1\left(p,J,\frac{4\pi}{3}+\theta,c',\varphi'\right).$$

Now we can combine these functions by using the procedure described in Section 2. This gives the expression $F(p,J,\theta,c',\varphi') = e^{\alpha\left(g_{\varphi'}(\theta,\alpha)-\frac{p+a}{J}\right)} = 1$, or, equivalently, $J = \dfrac{p+a}{g_{\varphi'}(\theta,\alpha)}$, where the function $g_{\varphi'}(\theta,\alpha)$ is defined by

$$g_{\varphi'}(\theta,\alpha) = \frac{1}{\alpha}\ln\left(e^{\alpha g(\theta)} + e^{\alpha g\left(\frac{\pi}{3}-\theta\right)} + e^{\alpha g\left(-\frac{\pi}{3}-\theta\right)} + e^{\alpha g(\pi-\theta)} + e^{\alpha g\left(\frac{2\pi}{3}+\theta\right)} + e^{\alpha g\left(\frac{4\pi}{3}+\theta\right)}\right).$$

Fig. 7 shows the shape of the approximation for $\varphi' = 15^{\circ}$ and for $\varphi' = 40^{\circ}$.

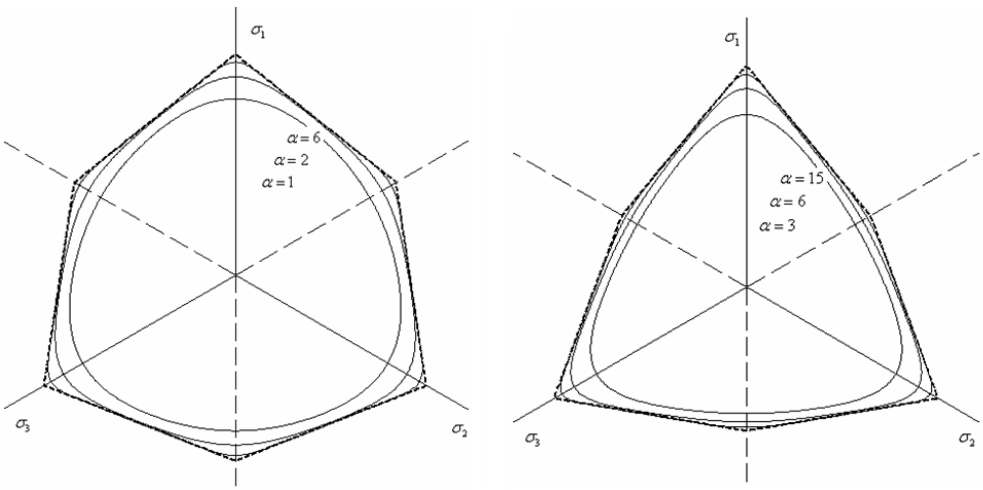

Figure 7: smooth approximations of MC surfaces for $\varphi' = 15^{\circ}$ and for $\varphi' = 40^{\circ}$.

Now, it is easy to modify this expression in order to smooth the vertex. By analogy with Subsection 3.1, we can construct the explicit equation $J = \dfrac{p+a}{g_{\varphi'}(\theta,\alpha)}\left(1-e^{-\beta\left(\frac{p+a}{\varepsilon}-1\right)}\right)^{\frac{1}{\delta}}$, $\delta > 1$, which corresponds to the implicit equation $\left|\dfrac{Jg_{\varphi'}(\theta,\alpha)}{p+a}\right|^{\delta} + e^{-\beta\left(\frac{p+a}{\varepsilon}-1\right)} = 1$. Other interesting choice for this case could be $2^{-\left(\frac{p+a}{\varepsilon}-1\right)}\cosh\dfrac{Jg_{\varphi'}(\theta,\alpha)\ln 2}{\varepsilon} = 1$.

3.3 Original Cam-Clay (OCC) surface

If we consider the general expression $\frac{JG_{\varphi'}(\theta)}{p_0+a}+\frac{p+a}{p_0+a}\ln\frac{p+a}{p_0+a}=0$, where $a=c'\cot\varphi'$ and $G_{\varphi'}(\theta)$ is a function that contains the influence of Lode's angle θ, then we can construct the smoothed surface $2e^{\alpha\frac{p+a}{p_0+a}\ln\frac{p+a}{p_0+a}}\cosh\frac{\alpha JH_{\varphi'}(\theta,\beta)}{p_0+a}=1$, where, for instance, $H_{\varphi'}(\theta,\beta)=C$ if $G_{\varphi'}(\theta)=C$ and $H_{\varphi'}(\theta,\beta)=g_{\varphi'}(\theta,\beta)$ if $G_{\varphi'}(\theta)=g(\theta)$. For a given absolute error ε_1 (Figure 4) we can define the dimensionless variable $c_1=1-\frac{\varepsilon_1}{p_0+a}$. Then, it holds that $\alpha=\frac{\ln 2}{c_1\ln\frac{1}{c_1}}$, which leads to the expression $2^{-\left(\frac{\frac{p+a}{p_0+a}\ln\frac{p+a}{p_0+a}}{c_1\ln c_1}-1\right)}\cosh\frac{JH^2_{\varphi'}(\theta,\beta_2)\ln 2}{(p_0+a)c_1\ln\frac{1}{c_1}}=1$ for the smoothed yield surface. It must been taken into account that this procedure also generates an absolute error ε_2 (Fig. 4) at point $p=-a$, $J=0$, though the original load surface is smooth there. If we define $c_2=\frac{\varepsilon_2}{p_0+a}$, then we have that, for a given $\alpha\geq e\ln 2$, c_1, c_2 are the two solutions of the equation $c\ln\frac{1}{c}=\frac{\ln 2}{\alpha}$.

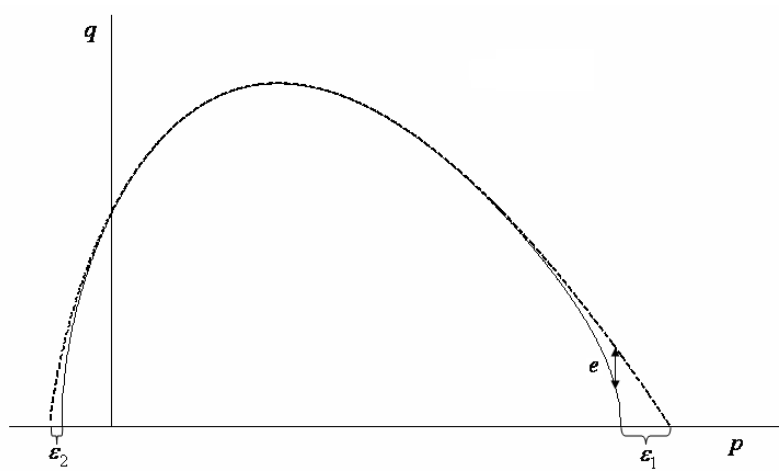

Figure 4: smooth approximation of the OCC surface.

4 MULTI-SURFACE SMOOTHING

The presented smoothing procedure allows us to reduce a multi-surface elasto-plastic model to an 'equivalent' single-surface elasto-plastic model. Let us consider for a multi-surface model all its k yield functions $F_1(\boldsymbol{\sigma},\boldsymbol{\chi}_1)$, ..., $F_k(\boldsymbol{\sigma},\boldsymbol{\chi}_k)$ and their corresponding plastic

potential functions and hardening functions, $G_1(\boldsymbol{\sigma},\xi_1)$, ..., $G_k(\boldsymbol{\sigma},\xi_k)$ and $\boldsymbol{\chi}_1\left(\boldsymbol{\varepsilon}^p\right)$, ..., $\boldsymbol{\chi}_k\left(\boldsymbol{\varepsilon}^p\right)$, respectively. Then, $d\boldsymbol{\varepsilon}^p = \sum_{i=1}^{n} \frac{\partial G_i}{\partial \boldsymbol{\sigma}} d\lambda_i$ and $d\boldsymbol{\chi}_i = \sum_{j=1}^{n} h_{ij} d\lambda_j$ are two general expressions for the flow rule and for the hardening rules, respectively. If we suppose that the equations of the plastic surfaces are $F_1(\boldsymbol{\sigma},\boldsymbol{\chi}_1)=1$, ..., $F_k(\boldsymbol{\sigma},\boldsymbol{\chi}_k)=1$, where $F_1(\boldsymbol{\sigma},\boldsymbol{\chi}_1)\geq 0$, ..., $F_k(\boldsymbol{\sigma},\boldsymbol{\chi}_k)\geq 0$, then, when all the yield functions are regular, we can construct the yield function $F\left(\boldsymbol{\sigma},\boldsymbol{\chi}_1,...,\boldsymbol{\chi}_k\right)=\sum_{i=1}^{k}\left(F_i(\boldsymbol{\sigma},\boldsymbol{\chi}_i)\right)^{\alpha_i}$ and the corresponding smooth single yield surface $F\left(\boldsymbol{\sigma},\boldsymbol{\chi}_1,...,\boldsymbol{\chi}_k\right)=1$. The flow rule $d\boldsymbol{\varepsilon}^p = d\lambda \sum_{i=1}^{k} w_i \frac{\partial G_i}{\partial \boldsymbol{\sigma}}$ can be used, where weights w_i are defined as in Subsection 2.3. This choice is motivated by the need of recovering the expression $d\boldsymbol{\varepsilon}^p = d\lambda \frac{\partial F}{\partial \boldsymbol{\sigma}}$ in associated plasticity. Let us observe that if we take $d\lambda_i = w_i d\lambda$, then we have $d\boldsymbol{\varepsilon}^p = \sum_{i=1}^{k} d\lambda_i \frac{\partial G_i}{\partial \boldsymbol{\sigma}}$. This expression is formally equal to the classical Koiter's one, but with this approach only an independent plastic multiplier there exists. In the same way, we can write $d\boldsymbol{\chi}_i = d\lambda \sum_{j=1}^{n} w_j h_{ij}$. The value of the plastic multiplier $d\lambda$ is obtained by means of the usual single-surface consistency condition $dF=0$.

4.1 MC with a MC-OCC cap model

Consider a bi-surface shear-volumetric plastic model with an OCC cap. The elastic domain corresponding to such a model is in general bounded (in the first sextant) by the yield surfaces $G^1_{\varphi'}(\theta) - \frac{p+a}{J} = 0$ (shear surface) and $\frac{G^2_{\varphi'}(\theta)}{\ln\frac{p_0+a}{p+a}} - \frac{p+a}{J} = 0$ (OCC cap). In these equations, the expression of the functions $G^1_{\varphi'}$ and $G^2_{\varphi'}$ can correspond to a DP model, a MC model or other models.

A first step towards the smooth 'equivalent' single-surface could consist in substituting functions $G^1_{\varphi'}, G^2_{\varphi'}$ by other functions $H^1_{\varphi'}, H^2_{\varphi'}$ -which can be obtained like it has been shown in previous Sections- and making the composition $e^{\alpha\left(H^1_{\varphi'}(\theta,\beta_1)-\frac{p+a}{J}\right)} + e^{\alpha\left(\frac{H^2_{\varphi'}(\theta,\beta_2)}{\ln\frac{p_0+a}{p+a}}-\frac{p+a}{J}\right)} = 1$, which leads to the explicit expression

$$J = \frac{p+a}{h_{\varphi'}\left(p,\theta,\alpha,\beta_1,\beta_2\right)}, \text{ where } h_{\varphi'}\left(p,\theta,\alpha,\beta_1,\beta_2\right) = \frac{1}{\alpha}\ln\left(e^{\alpha H^1_{\varphi'}(\theta,\beta_1)} + e^{\alpha\frac{H^2_{\varphi'}(\theta,\beta_2)}{\ln\frac{p_0+a}{p+a}}}\right).$$

After that, there are still two singular points on the hydrostatic axis at $p=-a$ and at $p=p_0$. They can be eliminated, for instance, by taking the new explicit expression $J=\frac{p+a}{h_{\varphi'}(p,\theta,\alpha,\beta_1,\beta_2)}\left[\left(1-e^{-\beta\left(\frac{1}{c_2}\frac{p+a}{p_0+a}-1\right)}\right)\left(1-e^{\beta\left(\frac{1}{c_1}\frac{p+a}{p_0+a}-1\right)}\right)\right]^{\frac{1}{\delta}}$, where $\delta>1$, $\beta>0$ and $c_1,\ c_2>0$; in these conditions, J nulls at $p_1=-a+c_2(p_0+a)$ and at $p_2=-a+c_1(p_0+a)$. The implicit equation that corresponds to this smoothed surface is $\left|\frac{Jh_{\varphi'}(p,\theta,\alpha,\beta_1,\beta_2)}{p+a}\right|^{\delta}+e^{-\beta\left(\frac{1}{c_2}\frac{p+a}{p_0+a}-1\right)}+e^{\beta\left(\frac{1}{c_1}\frac{p+a}{p_0+a}-1\right)}-e^{-\beta\frac{p+a}{p_0+a}\frac{c_1-c_2}{c_1c_2}}=1$.

Figure 9 shows, for instance, a smoothed MC & MC-OCC surface.

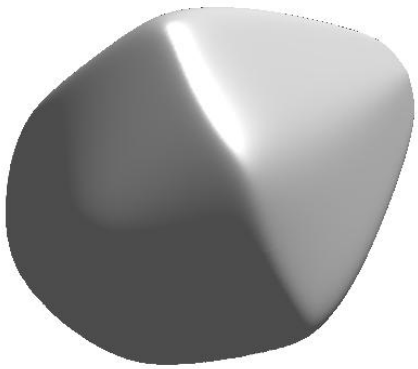

Figure 9: 3D representation of a regular approximation of a MC model with a MC-OCC cap.

4 CONCLUSIONS

We have described a smoothing technique based on the algebraic combination of different functions and we have showed how this technique can be used in the framework of isotropic plasticity to obtain smooth approximations for yield surfaces with corners. Moreover, the introduced procedures have been used to propose a reformulation of multi-surface plasticity. We have presented several application examples that illustrate the versatility of the presented approach.

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