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ON THE ASSOCIATIVITY OF THE DRUCKER-PRAGER MODEL

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1 INTRODUCTION

A plasticity model is said to be associated if the plastic strain flow is orthogonal to the loading surface, i.e. the elastic domain boundary. Using the language of convex analysis, associativity (or *normality*) means that it is possible to define the plastic strain flow as an element of the *sub-differential* of the *indicator function* of the elastic domain, which is a convex set in the stress space[1,2]. For the (perfectly-plastic) Drucker-Prager model, the flow is not in general orthogonal to the loading surface, hence it is classically considered as a non-associated model. However, in this standard definition, the loading surface does not depend on state variables, viz. the total and the plastic strain. In this contribution, making use of a loading surface *depending on state variables* (see e.g. [3]), it is shown that the non-associated Drucker-Prager model can be formulated as an associative model. However, this new loading surface no longer represents the elastic domain.

2 A NEW DESCRIPTION OF THE DRUCKER-PRAGER MODEL

In this Section, a plasticity model is defined by its Helmholtz free energy and its *pseudo-potential*[3]. Then, normality assumptions will be used to derive flow rules and it will be proved that these associated rules are equal to those of the non-associated (and perfectly-plastic) Drucker-Prager model. The state variables $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$ are assumed to belong to a convex subset H. Hence, the Helmholtz free energy is the sum of a smooth part ψ and of the *indicator function* I_H :

$$\Psi = \psi\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) + I_H\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) = \frac{1}{2}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p\right) : \mathbf{C} : \left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p\right) + I_H\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) \tag{1}$$

Under the assumption of isotropy, the elasticity tensor becomes $\mathbf{C} = (K - \frac{2}{3}G)\mathbf{1} \otimes \mathbf{1} + 2G\mathbf{I}$, where $\mathbf{1}$ is the second order identity tensor; $\mathbf{I}$ is the fourth order identity tensor; and $\otimes$ represents the tensor product. The term I_H is associated to (see also Figure 1a)

$$H = \left\{ (\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) \qquad \text{such that} \qquad \frac{3K\ tr\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p\right)}{3} - \frac{c}{\tan\varphi} := -h\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) \leq 0 \right\} \tag{2}$$

where φ is the friction angle, and c is the cohesion. The constraint defining H imposes that the elastic hydrostatic dilation $tr\left(\boldsymbol{\varepsilon}^e\right) = tr\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p\right)$ can never be greater that $\frac{c}{K\ \tan\varphi}$.

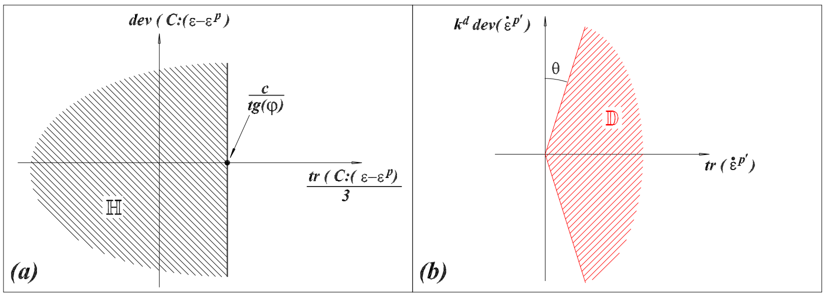

Figure 1: (a) Domain H of the admissible state variable values. (b) Domain D of the admissible fluxes.

The non-dissipative forces (nd) and the non-dissipative reaction forces (ndr) follow[3,4]

$$\begin{aligned}
&\sigma^{nd} = \tfrac{\partial\psi}{\partial\boldsymbol{\varepsilon}} = \mathbf{C} : \left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p\right) = K\ tr\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p\right)\mathbf{1} + 2G\ dev\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p\right) \\
&\tau^{nd} = \tfrac{\partial\psi}{\partial\boldsymbol{\varepsilon}^p} = -\mathbf{C} : \left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p\right) \\
&\left(\boldsymbol{\sigma}^{ndr}, \boldsymbol{\tau}^{ndr}\right) \in \partial I_H\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) = \begin{cases} (\mathbf{0}, \mathbf{0}) & & \text{for} \quad h\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) > 0 \\ (t\mathbf{1}, -t\mathbf{1}) & \text{with} \quad t \geq 0. & \text{for} \quad h\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) = 0 \\ \emptyset & & \text{for} \quad h\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) < 0 \end{cases}
\end{aligned} \tag{3}$$

A pseudo-potential depending on the flow $\dot{\boldsymbol{\varepsilon}}^p$ is introduced. It is a non-negative, convex function homogeneous of order 1. A pseudo-potential having these properties and the normality assumption lead to flow rules fulfilling the second principle of thermodynamics[3]:

$$\begin{aligned}
&\phi\left(\dot{\boldsymbol{\varepsilon}}', \dot{\boldsymbol{\varepsilon}}^{p\prime}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) = \phi_1\left(\dot{\boldsymbol{\varepsilon}}^{p\prime}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) = \phi_2\left(\dot{\boldsymbol{\varepsilon}}^{p\prime}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) + I_D\left(\dot{\boldsymbol{\varepsilon}}^{p\prime}\right) \\
&\phi_2 := \tfrac{c}{\tan\varphi} tr\left(\dot{\boldsymbol{\varepsilon}}^{p\prime}\right) + k_d\left(\tan\varphi - \tan\theta\right)\ h\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right)\ \left\|dev\left(\dot{\boldsymbol{\varepsilon}}^{p\prime}\right)\right\|
\end{aligned} \tag{4}$$

where $\tan\theta$ introduces the dilatancy effect, $k_d > 0$ is related to the friction angle φ [5] and

$$D = \left\{\dot{\boldsymbol{\varepsilon}}^{p\prime} \quad \text{such that} \quad f_D\left(\dot{\boldsymbol{\varepsilon}}^{p\prime}\right) := k_d \tan\theta\ \left\|dev\left(\dot{\boldsymbol{\varepsilon}}^{p\prime}\right)\right\| - tr\left(\dot{\boldsymbol{\varepsilon}}^{p\prime}\right) \leq 0\right\} \tag{5}$$

(see Figure 1b). The positivity of ϕ is ensured if $h\left(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) \geq 0$ (see the condition defining H) and if $0 \leq \tan\theta \leq \tan\varphi$. Note that the second term of ϕ_2 depends on $(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)$. The Legendre-Fenchel transform of ϕ_1 is equal to $\phi_1^*(\boldsymbol{\tau}^{d'}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) = I_{E_{(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)}}\left(\boldsymbol{\tau}^{d'}\right)$, $E_{(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p)} = \left\{\boldsymbol{\tau}^{d'} \ \text{ such that } \ f\left(\boldsymbol{\tau}^{d'}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) \leq 0\right\}$ and the *loading function* f is given by:

$$f\left(\boldsymbol{\tau}^{d'}; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) = \frac{\left\|dev\left(\boldsymbol{\tau}^{d\prime}\right)\right\|}{k_d} + \frac{tr\left(\boldsymbol{\tau}^{d\prime}\right)}{3}\tan\theta - c + \left(\tan\varphi - \tan\theta\right) K\ tr\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p\right) \tag{6}$$

where $\boldsymbol{\tau}^{d'}$ is the generic dissipative force associated to the plastic strain. Figures 2a,b illustrate two configurations of the set E associated to f. Observe that the last term in (6) introduces a translation of E along the trace axis depending on the instantaneous value of state variables. If $\theta = \varphi$ (associated case), then this translation is zero.

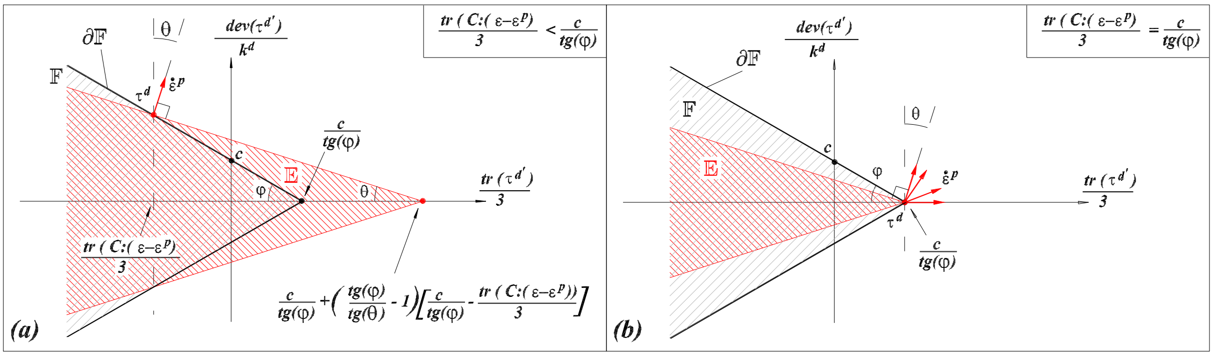

Figure 2: (a) The traditional elastic domain F and the set $E_{(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p)}$ for a generic value of the state variables $(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p) \in int(H)$. (b) The traditional elastic domain F and the set $E_{(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p)}$ when $\frac{tr(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p)}{3} = \frac{c}{\tan\varphi}$.

2.1 Normality conditions, *traditional* elastic domain and flow rules

The loading function defined in the previous section is different from the one usually adopted in the definition of elastic domain of the Drucker-Prager model. The relationships between them are investigated herafter. The normality rule applied on ϕ_1 reads:

$$\boldsymbol{\tau}^d \in \partial\phi_1\left(\dot{\boldsymbol{\varepsilon}}^p;\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p\right) = \partial\phi_2\left(\dot{\boldsymbol{\varepsilon}}^p;\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p\right) + \partial I_D\left(\dot{\boldsymbol{\varepsilon}}^p;\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p\right) \tag{7}$$

where $\dot{\boldsymbol{\varepsilon}}^p$ is the *actual* plastic strain flow and $\boldsymbol{\tau}^d = -\boldsymbol{\tau}^{nd} - \boldsymbol{\tau}^{ndr}$ is the associated *dissipative* thermodynamic force. One can prove that (7) is equivalent to

$$\begin{aligned} &\tfrac{1}{3}tr\left(\boldsymbol{\tau}^d\right) = \tfrac{c}{\tan\varphi} - \gamma \\ &dev\left(\boldsymbol{\tau}^d\right) = k_d\left[\gamma\tan\theta + (\tan\varphi - \tan\theta)\left(\tfrac{c}{\tan\varphi} - K\ tr\left(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p\right)\right)\right]\mathbf{k} \\ &\gamma\ f_D\left(\dot{\boldsymbol{\varepsilon}}^p\right) = 0, \qquad \gamma \geq 0, \qquad f_D\left(\dot{\boldsymbol{\varepsilon}}^p\right) \leq 0 \end{aligned} \tag{8}$$

where $\mathbf{k} \in \partial\left\|dev\left(\dot{\boldsymbol{\varepsilon}}^p\right)\right\|$, i.e. $\|\mathbf{k}\| \leq 1$ and $\mathbf{k} = dev\left(\dot{\boldsymbol{\varepsilon}}^p\right)/\left\|dev\left(\dot{\boldsymbol{\varepsilon}}^p\right)\right\|$ when $dev\left(\dot{\boldsymbol{\varepsilon}}^p\right) \neq \mathbf{0}$. These relationships hold *at the present state* (note that $\boldsymbol{\tau}^d$ and $\dot{\boldsymbol{\varepsilon}}^p$ are used instead of $\boldsymbol{\tau}^{d'}$ and $\dot{\boldsymbol{\varepsilon}}^{p'}$). Hence, knowing that $\boldsymbol{\tau}^d = -\boldsymbol{\tau}^{nd} - \boldsymbol{\tau}^{ndr}$ holds by definition, Eqs. $(8)_{1-2}$ can be simplified. First, one can prove that for both cases $(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p) \in int\,(H)$ and $(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p) \in \partial H$, one has $tr\left(\boldsymbol{\sigma}^{ndr}\right) = tr\left(\boldsymbol{\tau}^{ndr}\right) = 0$. Then, using (3) in Eqs. $(8)_{1-2}$, one obtains

$$\begin{aligned} &\tfrac{1}{3}tr\left(\boldsymbol{\tau}^d\right) = -\tfrac{1}{3}tr\left(\boldsymbol{\tau}^{nd}\right) = K\ tr\left(\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}^p\right) = \tfrac{c}{\tan\varphi} - \gamma \\ &dev\left(\boldsymbol{\tau}^d\right) = -dev\left(\boldsymbol{\tau}^{nd}\right) = \gamma k_d\tan\varphi\mathbf{k} \end{aligned} \tag{9}$$

with $\gamma = h(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p) \geq 0$. Observe that these relationships define a *circle* in the $\boldsymbol{\tau}^{d'}$−space, such that all its points $\boldsymbol{\tau}$ have the same trace, equal to $(3c/\tan\varphi \ - 3\gamma)$, and the maximum norm of the deviatoric part is equal to $\gamma k_d \tan\varphi$ (see also Figure 2). The parameter γ define both position and size of these sets. The union of all the circles, i.e. all $\gamma \geq 0$, is a *cone* F in the $\boldsymbol{\tau}^{d'}$−space, having an apex angle equal to φ. This set corresponds to the "traditional" elastic domain of the Drucker-Prager model[4]. Comparing this cone with $E_{(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p)}$, associated to the loading function f of Eq. (6), one can observe that (i) the plastic flow is orthogonal to $E_{(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p)}$; (ii) F is fixed in the $\boldsymbol{\tau}^{d'}$−space and (iii) given γ, the intersection of the two sets is the circumference delimiting the circle associated to the same γ value. The flow rules are normality conditions dual to the ones of Eq. (7):

$$\dot{\boldsymbol{\varepsilon}}^p \in \partial\phi_1^*\left(\boldsymbol{\tau}^d; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) = \partial I_{E_{(\boldsymbol{\varepsilon},\boldsymbol{\varepsilon}^p)}}\left(\boldsymbol{\tau}^d\right) \tag{10}$$

Equivalently,

$$\begin{aligned} &tr\left(\dot{\boldsymbol{\varepsilon}}^p\right) = \dot{\lambda}\tan\theta, \qquad dev\left(\dot{\boldsymbol{\varepsilon}}^p\right) = \dot{\lambda}\tfrac{1}{k_d}\mathbf{m} \\ &\dot{\lambda} f\left(\boldsymbol{\tau}^d; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) = 0, \qquad \dot{\lambda} \geq 0, \qquad f\left(\boldsymbol{\tau}^d; \boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p\right) \leq 0 \end{aligned} \tag{11}$$

where $\mathbf{m} \in \partial\left\|dev\left(\boldsymbol{\tau}^d\right)\right\|$. On the regular part of the loading surface, one has $\mathbf{m} = \frac{dev(\boldsymbol{\tau}^d)}{\|dev(\boldsymbol{\tau}^d)\|}$ and the consistency condition can be applied to compute $\dot{\lambda}$, accounting for the dependence of f on the state variables $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^p$. Conversely, at the apex it holds $\dot{\boldsymbol{\varepsilon}}^p = \dot{\boldsymbol{\varepsilon}}$, provided that $f_D(\dot{\boldsymbol{\varepsilon}}) \leq 0$. Eqs. (11) are the flow rules of a non-associated Drucker-Prager model.

3 CONCLUSIONS

An *associative* description of the non-associated Drucker-Prager model has been provided, by means of a loading function f having an additional dependence on state variables. The relationships between f and the "traditional" elastic domain have been discussed. Work is in progress to analyze the common points of this approach with the bipotential formulation[5].

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