VIII International Conference on Computational Plasticity
COMPLAS VIII
E. Oñate and D.R.J. Owen (Eds)


# PROPOSAL OF PARTICLE DISCRETIZATION SCHEME FOR SOLVING BOUNDARY VALUE PROBLEM OF CONTINUUM

**M.L.L. Wijerathne*, Kenji Oguni†, and Muneo Hori‡**

* † ‡ Earthquake Research Institue The University of Tokyo
Yayoi-1-1-1, Bunkyo-ku, Tokyo-113-0000, Japan
*e-mail: lalith@eri.u-tokyo.ac.jp

**Key words:** failure analysis, FEM with particle discretization, non-overlapping characteristic functions

**Summary.** *A novel numerical technique, FEM-$\beta$, with the potential of massive scale simulations of failure of bulk bodies is presented. FEM-$\beta$ obtains numerical solution for a variational problem equivalent to the BVP of a deformable continuum, using non-overlapping characteristic functions.*

## 1 INTRODUCTION

Being a numerical method for solving boundary value problems (BVPs), FEM is extensively used for various classes of problems in solid mechanics. However, in failure studies, FEM needs complicated treatments when the configuration changes due to crack progression. This complication is due to the overlapping set of characteristic functions used to discretize the displacement field. Joint element method, mesh-less method are some of such computationally expensive techniques. Rigid-body-spring modeling (RBSM) is another numerical method used for analysis of deformable bodies. Being a particle physics type method, RBSM models deformable body as an assembly of rigid bodies connected through springs, which can be viewed as discretization of displacement field using non-overlapping characteristic functions. Due to the inherent discontinuous nature, RBSM can easily model failure by sequential breakage of springs, which is computationally very efficient. However, RBSM is lacking of rigorous method to determine spring properties and the equivalence between the RBSM simulations and the physical phenomenon under consideration is not guaranteed since this is not a numerical method for BVPs.

The new method introduced in this paper, FEM-$\beta$[1], can be considered as a numerical technique with the merits of both FEM and RBSM. We introduce non-overlapping set of shape functions, which is the key for easy treatment of failure in RBSM, to FEM framework and obtain a numerical technique for solving BVPs of deformable bodies with RBSM type easy treatment of failure. These non-overlapping shape functions introduce displacement discontinuities to the model, almost everywhere. Some of these can be used to represent cracks. Crack or damage can be naturally represented by appropriately changing stiffness matrix components. Unlike FEM, FEM-$\beta$ does not require special treatments to deal with failure since it is inherited with a discontinuous nature of the displacement field. Therefore, compared with FEM, this method is computationally very efficient in failure studies. The formulation of FEM-$\beta$ and treatment of failure are briefly presented in the rest of this paper.

## 2 FORMULATION

As the simplest case, we consider the deformation of homogeneous linear elastic body $\Omega$ assuming quasi-static and infinitesimally small deformation. The equivalent BVP for this physical problem is posed as

$$\begin{cases} c_{ijkl}u_{k,li} = 0 & \text{in } \Omega \\ u_i = \bar{u}_i & \text{on } \partial\Omega \end{cases} \tag{1}$$

, where $c_{ijkl}$ is elastic tensor and $\bar{u}_i$ is prescribed boundary displacements. This BVP is transformed to an equivalent variational problem using following functional for displacement and stress.

$$I(\mathbf{u}, \boldsymbol{\sigma}) = \int_{\Omega} \sigma_{ij}\epsilon_{ij} - u_i b_i - \frac{1}{2} c^{-1}_{ijkl}\sigma_{ij}\sigma_{kl} ds \tag{2}$$

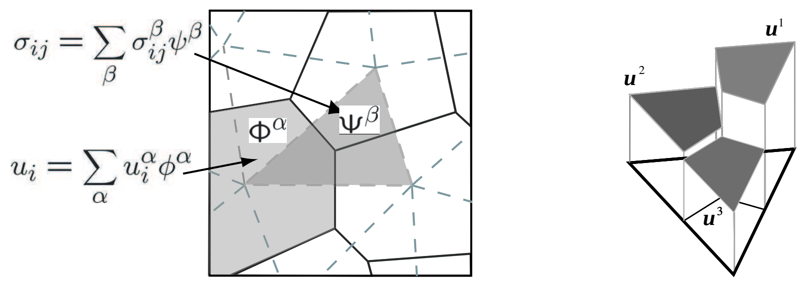
Here, $c^{-1}_{ijlk}$ is the inverse of $c_{ijlk}$ and $b_i$ is body force.

The discretization scheme used in FEM-$\beta$ is called particle discretization; field variables are discretized using a set of non-overlapping characteristic functions. Obviously, these discontinuous characteristic functions are not admissible for solving BVP. To address this, FEM-$\beta$ uses conjugate geometries for discretizing field variables and its derivatives. First a set of points $\{x^\alpha\}(\alpha = 1, 2, \ldots, N)$ are distributed and $\Omega$ is decomposed into $\Phi^\alpha$'s based on Voronoi diagrams for $\{x^\alpha\}$(see Fig 1). Then $u_i$ and body force, $b_i$ are decomposed using $\phi^\alpha$. $\phi^\alpha$ is the characteristic function on $\Phi^\alpha$ defined as $\phi^\alpha(\mathbf{x}) = 1$ for $\mathbf{x} \in \Phi^\alpha$and $=0$ for $\mathbf{x} \notin \Phi^\alpha$. Denoting the displacements and body force in $\Phi^\alpha$ as $u^\alpha_i$ and $b^\alpha_i$ respectively, $u_i$ and $b_i$ can be decomposed as $u_i(\mathbf{x}) = \sum_\alpha u^\alpha_i \phi^\alpha(\mathbf{x}),$ and $b_i(\mathbf{x}) = \sum_\alpha b^\alpha_i \phi^\alpha$.

For calculation of derivatives, we cannot use the above discretized form of $u_i$ since the derivative of $\phi^\alpha$ becomes a delta function along the boundary of $\Phi^\alpha$. However, we can calculate average of derivatives by taking the average over a domain that includes $\partial\Phi^\alpha$. Delaunay tessellation, $\{\Psi^\beta\}$, which is the conjugate geometry of Voronoi tessellation is used for this purpose. Based on this conjugate discretization $\sigma_{ij}$ can be discretized as

$$\sigma_{ij}(\mathbf{x}) = \sum_\alpha \sigma^\beta_{ij}\psi^\beta(\mathbf{x}) \tag{3}$$

, where $\{\psi^\beta\}$ are a non-overlapping set of characteristic functions for $\Psi^\beta$'s (see Fig. 1). By substituting above discretized forms in to $I$ and setting $\partial I/\partial\sigma^\beta_{ij} = 0$ an expression for $\sigma^\beta_{ij}$ can be obtained as $\sigma^\beta_{ij} = \sum_\alpha c_{ijkl}B^{\beta\alpha}_k u^\alpha_l$. This provides an expression for average




(a) (b)

Figure 1: a) Conjugate geometries, Voronoi and Delanay tessellations are used for discretizing displacements and derivatives . b) Displacement discontinuities at Voronoi blocks boundaries

strain as $\bar{\epsilon}_{ij}^{\beta} = \sum_\alpha B_i^{\beta\alpha} u_j^\alpha = \frac{u_j}{\Psi^\beta} \int_{\Psi^\beta} \phi_{,i}^\alpha(\mathbf{x}) ds$. This integral can be explicitly computed by transforming into a line integral using the Gauss theorem. Note that, unlike the usual FEM, FEM-$\beta$ obtains an expression for strain by minimizing $I$. Now, setting $\partial I / \partial u_i^\alpha = 0$ the following governing matrix equation of FEM-$\beta$ can be obtained.

$$\sum_{\alpha'} k_{ij}^{\alpha\alpha'} u_j^{\alpha'} - b_i^\alpha = 0$$

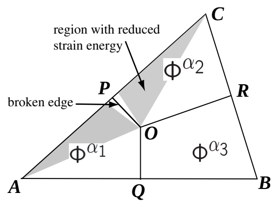
, where the stiffness matrix $k_{ij}^{\alpha\alpha'} = \sum_\beta \Psi^\beta c_{ijkl} B_k^{\beta\alpha} B_l^{\beta\alpha'}$. When Delaunay triangles are regarded as elements, the stiffness matrix of FEM-$\beta$ coincides with that of FEM. Consequently, nodal displacements calculated by FEM-$\beta$ is as accurate as that by FEM when triangular/tetrahedral elements are used in 2D/3D.

In addition to the translations, rotational degree of freedom (DOF) also is introduced to improve the accuracy of FEM-$\beta$. Even though rotational DOF has less effect under usual condition, it improves the accuracy in the presence of rapidly varying displacement field like crack tip displacement filed.

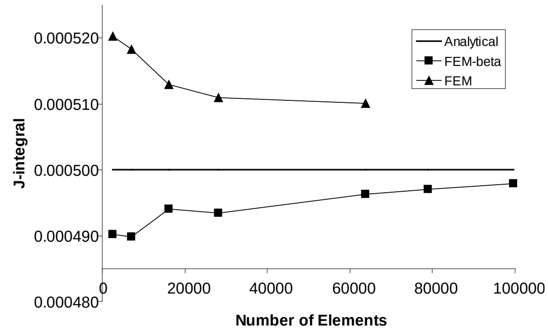
## 3 SPRING PROPERTIES AND EASY TREATMENT OF FAILURE

The discretized displacement field using non-overlapping shape functions over Voronoi blocks can be viewed as an assembly of rigid bodies connected with springs. The properties of these springs are given by the components of stiffness matrix. As an example, the springs properties connecting Voronoi blocks $\Omega^\alpha$ and $\Omega^{\alpha'}$ are given by the components of $k_{ij}^{\alpha\alpha'}$. As it was shown in the previous section, unlike in RBSM, these spring properties or components of $k_{ij}^{\alpha\alpha'}$ are rigorously determined using the elastic tensor, $c_{ijkl}$.

The non-overlapping shape functions, $\phi^\alpha$'s, introduce displacement discontinuities along Voronoi block boundaries (see Fig. 1 b). These discontinuities can be regarded as potential paths for cracks to propagate. Using a suitable failure criteria, crack growth can be




(a)




(b)

Figure 2: a) Failure modeling by breaking springs. b) Variation of $J$-integral for a mode-I crack in semi-infinite domain w.r.t. number of discretizations

easily modelled by simply breaking the springs between Voronoi blocks or appropriately changing stiffness matrix components. This numerically efficient treatment of failure is the major advantage of FEM-$\beta$.

As an example, if the springs across $PO$ between segments $\Phi^{\alpha_1}$ and $\Phi^{\alpha_2}$ are to be broken under some failure criterion, the components of $k_{ij}^{\alpha_1\alpha_2}$ should be appropriately reduced. This is equivalent to reduction of strain energy in segment $AOC$. The resulting crack tip stress field is always underestimated since we consider an average value for stress (see Eq. 3) in Delaunay triangle $ABC$ . To examine this approximate treatment, we consider a mode-I crack in a semi-infinite homogeneous body subjected to far field stress and calculated $J$-integral using both FEM and FEM-$\beta$. From Fig. 2a, it is clear that FEM-$\beta$ underestimates strain energy and has upper bound while FEM overestimates and has a lower bound. The higher accuracy of FEM-$\beta$ solution is due to the presence of rotational degree of freedom.

## 4 CONCLUSIONS

A new method for simulating failure of deformable bodies is proposed. FEM-$\beta$ is numerical method for solving BVP which provides RBSM type easy treatment of failure. This numerically efficient failure treatment makes FEM-$\beta$ a prominent candidate for analysing large scale BVPs involving failure (e.g. fault propagation in earth crust).

## REFERENCES

[1] Hori M, Oguni K, Sakaguchi H. Proposal of FEM implemented with particle discretization for analysis of failure phenomena. *J. of Mech. and Phys. Solids*, **53 (3)**, 681-703, MAR 2005