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MATERIAL BIFURCATION ANALYSIS

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Summary. *The main aim of this document is to obtain general explicit expressions for the critical failure direction and the critical hardening modulus corresponding to the best-known classical continuum constitutive models (namely, continuum damage and plasticity models). To reach this aim, the ellipticity condition of the constitutive tangent operator will play a determinant role.*

1 INTRODUCTION

Consider the Tangent Acoustic Tensor ($\mathbf{Q}(\mathbf{N})$) defined as $\mathbf{Q}(\mathbf{N}) = \left(\mathbf{N} \cdot \mathbb{C}^{in} \cdot \mathbf{N}\right)$, where $\mathbb{C}^{in}$ is the tangent operator and $\mathbf{N}$ is a vector normal to the localized band. The lost of ellipticity will take place ([1]) when the following condition is reached:

$$\det[\mathbf{Q}(\mathbf{N})] = 0 \tag{1}$$

This condition is the point of departure to obtain the critical values.

2 MATERIAL BIFURCATION CONDITIONS

Let us consider a material which behavior is described by a constitutive model characterized by a constitutive tensor, $\mathbb{C}^{in}$ (or tangent material operator), which expression reads:

$$\mathbb{C}^{in} = \xi \mathbb{C}^{e} - \mathcal{K}\left(\mathbb{C}^{e} : \mathbf{m} \otimes \mathbf{n} : \mathbb{C}^{e}\right) \tag{2}$$

with $\xi = \frac{q}{r} = (1-d)$, $\mathcal{K} = \frac{q(r) - \mathcal{H}^{d} r}{r^{3}}$ for damage models and $\mathcal{K} = \frac{1}{\mathcal{H}^{p} + \mathbf{n} : \mathbb{C}^{e} : \mathbf{m}}$ $\xi = 1$, for plasticity models. $\mathbf{n}$ is the flow plastic, $\mathbf{m}$ is the flow of the plastic potential, q is the stress-like hardening/softening variable, d is a damage variable whose value ranges from 0 to 1, r is the internal variable, see [2] for more details.

Applying the definition of the standard fourth-order isotropic elastic modulus tensor in function of Lamé's parameters (λ, μ), *i.e.:* $\mathbb{C}^{e} = 2\mu\mathbb{I} + \lambda(\mathbf{1} \otimes \mathbf{1})$, and some considerations (see [3], [4], [5]), equation (1) can be reduced to the following one:

$$\frac{\xi}{\mathcal{K}(\mathcal{H})} = Z(\mathbf{N}) \tag{3}$$

with
$$Z(\mathbf{N}) = \lambda^2 \mathrm{Tr}(\mathbf{m})\mathrm{Tr}(\mathbf{n})(a+b) + 2\lambda\mu\mathrm{Tr}(\mathbf{n})(\mathbf{N}\cdot\mathbf{m}\cdot\mathbf{N})(a+b) + \\ + 2\lambda\mu\mathrm{Tr}(\mathbf{m})(\mathbf{N}\cdot\mathbf{n}\cdot\mathbf{N})(a+b) + 4\mu^2 a(\mathbf{N}\cdot\mathbf{n}\cdot\mathbf{N}\cdot\mathbf{m}) + \\ + 4\mu^2 b(\mathbf{N}\cdot\mathbf{m}\cdot\mathbf{N})(\mathbf{N}\cdot\mathbf{n}\cdot\mathbf{N}) \tag{4}$$

where $a = \frac{1}{\mu}$, $b = -\frac{(\lambda+\mu)}{(\lambda+2\mu)}\frac{1}{\mu}$.

The problem to be solved here is to find the critical normal vector $\mathbf{N}_{crit}$, which can be done by maximizing function (4). Once we have obtained the critical values $\mathbf{N}_{crit}$, we can obtain $\mathcal{H}_{crit}$ by substituting $\mathbf{N}_{crit}$ in equation (3).

3 CRITICAL VALUES

3.1 Case of colinearity between $\mathbf{n}$ and $\mathbf{m}$

For the coaxial non-associated case the principal directions for $\mathbf{n}$ coincide with the principal directions of $\mathbf{m}$ but $\mathbf{n} \neq \mathbf{m}$.

We can explicitly express the corresponding angles as:
$$\tan^2\theta_{crit} = \frac{[(\mathrm{m}_3-\mathrm{m}_1)\mathrm{n}_2 + (\mathrm{n}_3-\mathrm{n}_1)\mathrm{m}_2]\nu + (2\mathrm{n}_3-\mathrm{n}_1)\mathrm{m}_3 - \mathrm{m}_1\mathrm{n}_3}{[(\mathrm{m}_1-\mathrm{m}_3)\mathrm{n}_2 + (\mathrm{n}_1-\mathrm{n}_3)\mathrm{m}_2]\nu + (2\mathrm{n}_1-\mathrm{n}_3)\mathrm{m}_1 - \mathrm{n}_1\mathrm{m}_3} \tag{5}$$
where ν is the Poisson's ratio. It is interesting to observe that the critical angle does not depend on the Young's modulus E.

3.2 The critical Hardening/Softening parameter ($\mathcal{H}_{crit}$)

Damage Models

For the isotropic damage case $\mathcal{H}^d_{crit}$ is given by:
$$\mathcal{H}^d_{crit} = \xi\left(1 - \frac{r^2}{Z_{\max}(\mathbf{N})}\right) \tag{6}$$
where $Z_{\max}(\mathbf{N})$ is the maximum value from (4).

Plasticity Models
$$\mathcal{H}^p_{crit} = \frac{E}{4(1-\nu^2)}\left\{\frac{[(\mathrm{m}_{1-3})(\nu\mathrm{n}_2+\mathrm{n}_1)+(\mathrm{n}_{1-3})(\nu\mathrm{m}_2+\mathrm{m}_1)]^2}{(\mathrm{m}_{1-3})(\mathrm{n}_{1-3})} - 4[\nu(\mathrm{m}_2\mathrm{n}_1+\mathrm{m}_1\mathrm{n}_2)+(\mathrm{m}_2\mathrm{n}_2+\mathrm{m}_1\mathrm{n}_1)]\right\} \tag{7}$$
where $\mathrm{m}_{1-3} \equiv \mathrm{m}_1 - \mathrm{m}_3$; $\mathrm{n}_{1-3} \equiv \mathrm{n}_1 - \mathrm{n}_3$.

4 CRITICAL VALUES FOR SOME CONSTITUTIVE MODELS

Several classic models of plasticity and damage are employed

4.1 One parameter models

	CRITERIA		
	RANKINE	VON MISES	TRESCA
critical angle	$\tan^2\theta_{crit}=0\Rightarrow\begin{cases}\theta_1=0\\ \theta_2=0\end{cases}$	$\tan^2\theta_{crit}=-\dfrac{s_3+\nu s_2}{s_1+\nu s_2}$	$\tan^2\theta_{crit}=1\Rightarrow\begin{cases}\theta_1=+45^{o}\\ \theta_2=-45^{o}\end{cases}$
critical herdening modulus	$\mathcal{H}^p_{crit}=0$	$\mathcal{H}^p_{crit}=-\dfrac{3E s_2^2}{2\left(s_1^2+s_2^2+s_3^2\right)}$	$\mathcal{H}^p_{crit}=0$

where s_i are the principal values of the deviatoric stress tensor $\mathbf{s}$.

4.2 Two parameter models

CRITICAL VALUES FOR THE MOHR-COULOMB CRITERION	
critical angle	$\tan^2\theta_{crit}=-\dfrac{2\sin\psi\sin\phi+\sin\phi+\sin\psi}{2\sin\psi\sin\phi-\sin\phi-\sin\psi}$
critical hardening modulus	$\mathcal{H}^p_{crit}=\dfrac{E}{8(1-\nu^2)}\left[\dfrac{(\sin\psi-\sin\phi)^2}{\sqrt{\left(1+\sin^2\psi\right)\left(1+\sin^2\phi\right)}}\right]$

where ϕ is the angle of internal friction and ψ is the dilatancy angle.

CRITICAL VALUES FOR THE MOHR CRITERION (ASSOCIATED CASE)	
critical angle	$\tan^2\theta_{crit}=\dfrac{2-\sin\phi-\sin\psi}{2+\sin\phi+\sin\psi}$
critical hardening modulus	$\mathcal{H}^p_{crit}=\dfrac{E}{4(1-\nu^2)}\dfrac{(N'-M')^2}{(1+M')(1+N')}$

where $N'=\dfrac{1-\sin\phi}{1+\sin\phi}$; $M'=\dfrac{1-\sin\psi}{1+\sin\psi}$ with $N'\geq 0$, $M'\geq 0$

CRITICAL VALUES FOR THE DRUCKER-PRAGER CRITERION (ASSOCIATED CASE)	
critical angle	$\tan^2\theta_{crit}=\dfrac{-(1+\nu)(\alpha_1\alpha_4+\alpha_2\alpha_3)-2(s_3+\nu s_2)\alpha_1\alpha_2}{(1+\nu)(\alpha_1\alpha_4+\alpha_2\alpha_3)+2(s_1+\nu s_2)\alpha_1\alpha_2}$
critical hardening modulus	$\mathcal{H}^p_{crit}=\dfrac{E}{(1-\nu^2)}[A_1+A_2+A_3]$

For the Drucker-Prager criterion it was considered $\mathbf{n}=\alpha_1\mathbf{s}+\alpha_3\mathbf{1}$ and $\mathbf{m}=\alpha_2\mathbf{s}+\alpha_4\mathbf{1}$.

$$A_1 = \frac{[(1-2\nu)s_3\alpha_2 + (1+\nu)\alpha_4][(1-2\nu)s_3\alpha_1 + (1+\nu)\alpha_3]}{(1-2\nu)},$$

$$A_2 = \frac{[2(\nu s_2 + s_1)\alpha_1\alpha_2 + (1+\nu)(\alpha_1\alpha_4 + \alpha_2\alpha_3)]^2}{4\alpha_1\alpha_2}, \quad A_3 = -3(1-\nu)\frac{(1-2\nu)\tau_{oct}^2\alpha_1\alpha_2 + (1+\nu)(\alpha_3\alpha_4)}{(1-2\nu)},$$

with $\tau_{oct}^2 = \frac{1}{9}\left[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2\right]$.

4.3 Isotropic Damage Model

ISOTROPIC DAMAGE MODEL	
critical angle	$\tan^2\theta_{crit} = -\dfrac{\varepsilon_3 + \nu\varepsilon_2}{\varepsilon_1 + \nu\varepsilon_2}$
critical hardening modulus	$\mathcal{H}_{crit}^d = (1-d)\left[1 - \dfrac{(\lambda+\mu)r^2}{\left\{\lambda^2(\mathrm{Tr}(\boldsymbol{\varepsilon}))^2 + \left[(\varepsilon_1-\varepsilon_3)^2 + 2\mathrm{Tr}(\boldsymbol{\varepsilon})(\varepsilon_1+\varepsilon_3)\right]\lambda\mu + 2\mu^2\left(\varepsilon_1^2+\varepsilon_3^2\right)\right\}}\right]$

5 CONCLUSIONS

Starting from the Tangent Acoustic Tensor we obtained explicit formulae for the critical hardening modulus and the normal to the critical plane at localization.

REFERENCES

[1] J.W. Rudnicki and J.R. Rice "Condition for the localization of the deformation in pressure-sensitive dilatant material". *J. Mech. Phys. Solids*, 23, 371-394, (1975).

[2] J. Oliver, M. Cervera, S. Oller and J. Lubliner. Isotropic damage models and smeared crack analysis of concrete. In N. Bićanić *et al.* (ed) Proc.. SCI-C Computer Aided Analysis and Design of Concrete Structures, pp. 945-957. (1990).

[3] D. Bigoni and D. Zaccaria, "On strain localization analysis of elastoplastic materials at finite strains". *Int. J. Plasticity*, 9 N° 1, pp. 21-33, (1993).

[4] D. Bigoni and T. Hueckel. "Uniqueness and localization – I. Associative and non-associative elastoplasticity". *Int. J. Solids Struct.*, 28(2), pp. 197-213, (1991).

[5] E.W.V. Chaves. *A three dimensional setting for strong discontinuities modelling in failure mechanics*. PhD Thesis, Technical University of Catalonia, Barcelona, Spain, (2002).