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ELASTOPLASTICITY BY MATHEMATICAL PROGRAMMING METHODS

K. Krabbenhøft*, A.V. Lyamin*, S.W. Sloan*, and P. Wriggers†

*Geotechnical Research Group
University of Newcastle, NSW, Australia
e-mail: {kristian.krabbenhoft,andrei.lyamin,scott.sloan}@newcastle.edu.au

†Institut für Baumechanik und Numerische Mechanik
Universitt Hannover, D–30167, Hannover, Germany
e-mail: wriggers@ibnm.uni-hannover.de

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Summary. *In the last 10–15 years a number of very powerful methods for general convex programming have been developed. Commonly labeled interior–point (IP) methods, these algorithms make it possible to solve a wide variety of practical, large-scale problems with moderate computational effort. One such class of problems is classical small-displacement, rate-independent elastoplasticity. In this paper we investigate the prospects of applying the IP methodology to this class of problems. In addition to applying standard IP otimizers, we also develop a slightly modified IP method which in term of generality, efficiency, and robustness appears to be fully competitive with conventional methods.*

1 MATHEMATICAL PROGRAMMING FORMULATION

Following the variational framework developed by Simo *et al.* [1] and applying a standard mixed stress–displacement finite element discretization, the problem of incremental linear elastic–perfectly plastic analysis can be written as

$$\begin{aligned} &\min_{\mathbf{u}_{n+1}} \max_{\boldsymbol{\sigma}_{n+1}} \; : \quad -\tfrac{1}{2}(\boldsymbol{\sigma}_{n+1}-\boldsymbol{\sigma}_n)^{\mathsf{T}}\mathbf{S}(\boldsymbol{\sigma}_{n+1}-\boldsymbol{\sigma}_n)+\boldsymbol{\sigma}_{n+1}\mathbf{B}(\mathbf{u}_{n+1}-\mathbf{u}_n)-\mathbf{p}^{\mathsf{T}}(\mathbf{u}_{n+1}-\mathbf{u}_n) \\ &\text{subject to} \; : \quad f(\boldsymbol{\sigma}^j_{n+1}) \le 0\,, \; j=0,\ldots,N_Y \end{aligned} \tag{1}$$

where $\boldsymbol{\sigma}$ are the stresses, $\mathbf{u}$ are the displacements, N_Y is the number of yield points (typically the Gauss points) and subscripts n and $n+1$ refer to the known and new states respectively. Furthermore, the matrices $\mathbf{S}$ and $\mathbf{B}$ and the vector $\mathbf{p}$ are given by

$$\mathbf{S}=\int_\Omega \mathbf{N}_\sigma^{\mathsf{T}}\mathbb{E}^{-1}\mathbf{N}_\sigma \,\mathrm{d}\Omega\,, \quad \mathbf{B}=\int_\Omega \mathbf{N}_\sigma^{\mathsf{T}}\boldsymbol{\nabla}\mathbf{N}_u \,\mathrm{d}\Omega\,, \quad \mathbf{p}=\int_\Omega \mathbf{N}_u^{\mathsf{T}}\mathbf{b}\,\mathrm{d}\Omega+\int_{\Gamma_\sigma}\mathbf{N}^{\mathsf{T}}\hat{\mathbf{t}}\,\mathrm{d}\Gamma \tag{2}$$

where $\mathbf{N}_\sigma$ and $\mathbf{N}_u$ are the stress and displacement shape functions respectively and $\mathbb{E}$ is the elastic constitutive matrix.

The min–max problem (1) can be transformed into a max problem by first solving with respect to the displacements. The resulting mathematical program is

$$\begin{aligned} &\text{maximize} && -\tfrac{1}{2}(\boldsymbol{\sigma}_{n+1}-\boldsymbol{\sigma}_n)^{\mathsf{T}}\mathbf{S}(\boldsymbol{\sigma}_{n+1}-\boldsymbol{\sigma}_n) \\ &\text{subejct to} && \mathbf{B}^{\mathsf{T}}\boldsymbol{\sigma}_{n+1}=\mathbf{p} \\ & && \mathbf{f}(\boldsymbol{\sigma}^j_{n+1})\leq\mathbf{0} \end{aligned} \tag{3}$$

where all yield inequalities have been collected in $\mathbf{f}(\boldsymbol{\sigma})$.

2 SOLUTION ALGORITHM

The basis of the solution algorithm applied to (3) is the so–called primal–dual interior point method, see e.g. [2] and references therein, which is the prototype of most modern convex programming algorithms. The fundamental approach is as follows. First convert the yield inequalities into equalities by addition of positively restricted slack variables $\mathbf{z}$. Then add a penalty term in form of the so–called logarithmic barrier function to the objective function. This avoids making explicit reference to the positivity requirements on the slack variables and has, in addition, a number of more profound consequences which, however, shall not be discussed here. The resulting problem then reads

$$\begin{aligned} &\text{maximize} && -\tfrac{1}{2}(\boldsymbol{\sigma}_{n+1}-\boldsymbol{\sigma}_n)^{\mathsf{T}}\mathbf{S}(\boldsymbol{\sigma}_{n+1}-\boldsymbol{\sigma}_n)+\textstyle\sum_{j=1}^{N_Y}\mu\log z^j \\ &\text{subejct to} && \mathbf{B}^{\mathsf{T}}\boldsymbol{\sigma}_{n+1}=\mathbf{p} \\ & && \mathbf{f}(\boldsymbol{\sigma}_{n+1})+\mathbf{z}_{n+1}=\mathbf{0} \end{aligned} \tag{4}$$

where μ is an arbitrarily small, positive constant.
The Karush–Kuhn–Tucker conditions associated with (4) are

$$\begin{aligned} \mathbf{r}_{\text{ss}} &= \mathbf{B}(\mathbf{u}_{n+1}-\mathbf{u}_n)-\mathbf{S}(\boldsymbol{\sigma}_{n+1}-\boldsymbol{\sigma}_n)-\nabla\mathbf{f}_{n+1}\boldsymbol{\lambda}_{n+1}=\mathbf{0} \\ \mathbf{r}_{\text{eq}} &= \mathbf{B}^{\mathsf{T}}\boldsymbol{\sigma}_{n+1}-\mathbf{p}=\mathbf{0} \\ \mathbf{r}_{\text{yc}} &= \mathbf{f}_{n+1}+\mathbf{z}_{n+1}=\mathbf{0} \\ \mathbf{r}_{\text{cs}} &= \mathbf{Z}_{n+1}\boldsymbol{\lambda}_{n+1}-\mu\mathbf{e}=\mathbf{0} \end{aligned} \tag{5}$$

where λ_{n+1} and $(\mathbf{u}_{n+1}-\mathbf{u}_n)$ are the Lagrange multipliers. These conditions are immediately seen to be the discrete governing equations for the problem considered.
The basic idea is now to solve these equations using Newton's method with a reduced step length to ensure that all positively restricted variables remain positive throughout the iterations.
Generally speaking, the difficulty in solving the optimality conditions associated with general mathematical programs is that usually no, or very little, knowledge of the final solution is available. It is well-known that Newton's method guarantees convergence only

if the initial point lies in the proximity of the solution, i.e. within the 'convergence radius'. For general optimization problems such a point is impossible to identify and the interior point methods attempt to address this fact by the addition of a barrier function to the objective and by using reduced step sizes. This usually results in a steady, but not particularly rapid, convergence. In order to improve the convergence characteristics and take advantage of the fact that for problems of elastoplastic analysis a reasonable estimate of the solution is available, an alternative, and very simple, strategy has been developed. The basic features of the interior point methodology are maintained, but instead of using a damped step size a full Newton step size is always used in conjunction with a very aggressive barrier reduction strategy. In this way some of the variables which are restricted to be positive, for example the plastic multipliers, may become negative. Therefore, after each update of the variables, infeasible points are projected back onto the boundary of the feasible solution space. Although very simple to implement this strategy has proved to be extremely powerful as is illustrated in the next Section.

3 REPRESENTATIVE RESULTS

We consider the perforated square plate shown in Figure 1. The material parameters are the following: Young's modulus: $E = 1000$, Poisson's ratio: $\nu = 0.3$, and uniaxial yield strength: $\sigma_Y = 1$. The state of deformation is plane and yielding is governed by the von Mises criterion. The plate is discretized by 2304 quadratic displacement elements (9506 displacement degrees of freedom). Ten equally sized displacement increments of magnitude $\Delta u = 0.001$ are applied as indicated in Figure 1. The resulting load–displacement curve is shown in Figure 2.

In Table 1 the solution statistics are shown. It is noted that the rate of convergence becomes quadratic as the solution is approached. Furthermore, as expected, the most iterations are used in the steps that bring about the greatest changes in the structural response, i.e. steps 4, 5, and 6. In these steps the norm of the residual can be observed to oscillate somewhat before converging. This indicates that a line search procedure would be a useful supplement although it does not seem to be strictly necessary. Finally, it should be mentioned that these oscillations disappear when the magnitude of the load step is decreased. The cost of each iteration is the same as with most conventional methods and a direct comparison can therefore be made in terms of the total number of iterations. All in all, the behaviour is quite satisfactory and the same trend has been observed for a large number of other problems with constitutive laws similar to the one used in this example.

REFERENCES

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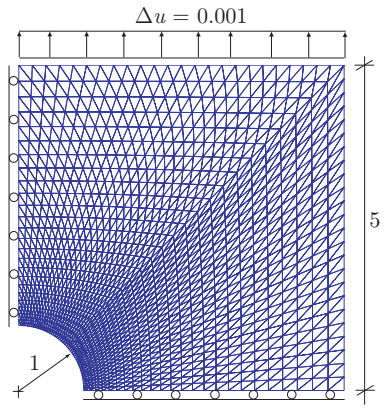

Figure 1: Perforated square plate, 2304 linear strain triangles, 9506 displacement degrees of freedom.

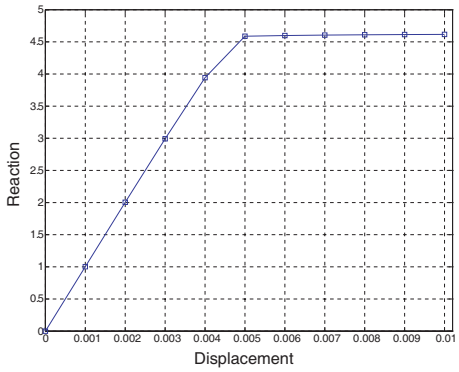

Figure 2: Load–displacement curve.

Iter/**React**	**1.0021**	**2.0042**	**2.9923**	**3.9427**	**4.5876**	**4.5985**	**4.6052**	**4.6099**	**4.6134**	**4.6161**
1	1.0e+00	1.0e+00	1.0e+00	1.0e+00	1.0e+00	1.0e+00	1.0e+00	1.0e+00	1.0e+00	1.0e+00
2	1.7e+02	2.0e+01	4.6e+02	4.1e+02	9.7e+02	4.0e+01	1.7e+01	1.1e+01	7.9e+00	6.6e+00
3	1.2e+03	2.2e–02	6.6e+01	8.9e+01	1.3e+02	1.0e+01	1.6e+00	1.1e+00	7.9e–01	3.8e–01
4	6.2e–07	1.1e–05	1.4e+01	7.0e+00	4.9e+01	3.6e+01	1.1e+00	1.6e–01	1.4e–01	1.0e–02
5	2.9e–12	9.1e–13	4.6e–02	1.7e–01	3.2e+01	1.0e+01	2.5e–04	2.5e–05	1.4e–03	1.3e–03
6			1.3e–05	1.3e+00	5.9e+01	2.6e+00	4.1e–06	5.8e–12	6.9e–06	4.1e–06
7			3.1e–06	6.7e–02	2.4e+01	2.9e–01	1.4e–12		2.1e–12	1.4e–12
8			9.6e–13	4.6e–01	4.4e+00	8.7e–03				
9				4.4e–04	1.3e+00	3.5e–07				
10				7.9e–02	3.2e–02	1.4e–12				
11				7.6e–05	7.6e–07					
12				1.2e–12	1.3e–12					

Table 1: Norms of total residual (5) as function of iteration number and load level.

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