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STATIC DEFORMATIONS AND NATURAL FREQUENCIES OF FUNCTIONALLY GRADED PLATES USING A HIGHER-ORDER THEORY AND A MESHLESS METHOD

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1 INTRODUCTION

An advantage of a plate made of a functionally graded material (FGM) over a laminated plate is that material properties vary continuously in a FGM but are discontinuous across adjoining layers in a laminated plate. Here we analyse a FG plate with material properties varying only in the thickness direction. In this paper we use the asymmetric collocation method with multiquadrics basis functions and a higher-order (HSDT) shear deformation theory to find static deformations and natural frequencies of square FG plates of various aspect ratios. This method was also used by Ferreira et al.[1] to study static deformations of FG plates. An advantage of this method over the finite element method (FEM) is that being a truly meshless method, the discretization of the domain is simple, both in 2D and 3D domains.

2 THE FINITE POINT MULTIQUADRIC METHOD

Consider the following linear elliptic boundary-value problem defined on a smooth domain Ω:

$$Lu(x) = s(x), x \in \Omega; \quad Bu(x) = f(x), x \in \partial\Omega \tag{1}$$

where $\partial\Omega$ is the boundary of Ω, L and B are linear differential operators, and s and f are smooth functions defined on Ω and $\partial\Omega$ respectively. We select N_B points ($\mathbf{x}^{(j)}, j = 1,\ldots,N_B$) on $\partial\Omega$ and ($N - N_B$) points ($\mathbf{x}^{(j)}, j = N_B+1, N_B+2\ldots,N$) in the interior of Ω. Let

$$u^h(x) = \sum_{j=1}^{N} a_j g\left(\left\|x - x^j\right\|, c\right) \tag{2}$$

be an approximate solution of the boundary-value problem where $a_1, a_2, \ldots, a_N$ are constants to be determined, $\left\| x - x^{(j)} \right\|$ is the Euclidean distance between points x and $x^{(j)}$, c is a constant, and g is a function of $\left\| x - x^{(j)} \right\|$ and c. Substitution from (2) into (1) and evaluating the resulting form of equations $(1)_2$ at the N_B points $\mathbf{x}^{(j)}, j = 1, \ldots, N_B$ and of equations $(1)_1$ at ($N - N_B$), points $\mathbf{x}^{(j)}, j = N_B + 1, N_B + 2 \ldots, N$ gives the following N algebraic equations for the determination of $a_1, a_2, \ldots, a_N$

$$\begin{aligned} &\sum_{j=1}^{N} a_j Lg\left(\left\| x - x^{(j)} \right\|, c\right)\Big|_{x=x^i} = s(x^{(i)}), i = N_B + 1, N_B + 2, \ldots N \\ &\sum_{j=1}^{N} a_j Bg\left(\left\| x - x^{(j)} \right\|, c\right)\Big|_{x=x^i} = f(x^{(i)}), i = 1, 2, \ldots N_B \end{aligned} \tag{3}$$

Depending upon the value of the parameter c and the form of function g, the set of equations (3) that determines $a_1, a_2, \ldots, a_N$ may become ill-conditioned. Also, the computational effort involved in solving (3) for $a_1, a_2, \ldots, a_N$ varies with the choice of the function g. Once equations (3) have been solved for a_j, then the approximate solution of the problem is given by (2).

3 THIRD-ORDER SHEAR DEFORMATION PLATE THEORY

The displacement field in the TSDT is given by

$$\begin{aligned} &u(x, y, z) = u_0(x, y) + z\phi_x(x, y) - c_1 z^3 \left(\phi_x(x, y) + \frac{\partial w(x, y)}{\partial x} \right) \\ &v(x, y, z) = v_0(x, y) + z\phi_y(x, y) - c_1 z^3 \left(\phi_y(x, y) + \frac{\partial w(x, y)}{\partial y} \right) \\ &w(x, y, z) = w_0(x, y) \end{aligned} \tag{4}$$

where $c_1 = 4/\left(3h^2\right)$, h is the plate thickness, z is the coordinate in the thickness direction, and the xy plane of the rectangular Cartesian coordinate system is located in the midplane of the plate. Functions ϕ_x and ϕ_y describe rotations about the x- and the y-axes of a line that is along the normal to the midsurface of the plate, u_0, v_0 and w_0 give displacements of a point on the midsurface of the plate along the x-, y- and z-axes respectively. Equations for the plate theory are derived by using the dynamic version of the principle of virtual work. That is,

$$
\begin{aligned}
&\textbf{a)}\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}=I_0\frac{\partial^2 u_0}{\partial t^2}+J_1\frac{\partial^2\phi_x}{\partial t^2}-c_1I_3\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial x}\right);\quad \textbf{b)}\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}=I_0\frac{\partial^2 v_0}{\partial t^2}+J_1\frac{\partial^2\phi_y}{\partial t^2}-c_1I_3\frac{\partial^2}{\partial t^2}\left(\frac{\partial w_0}{\partial y}\right)\\
&\textbf{c)}\frac{\partial \overline{Q}_x}{\partial x}+\frac{\partial \overline{Q}_y}{\partial y}+c_1\left(\frac{\partial^2 P_{xx}}{\partial x^2}+2\frac{\partial^2 P_{xy}}{\partial x\partial y}+\frac{\partial^2 P_{yy}}{\partial y^2}\right)+q==I_0\frac{\partial^2 w_0}{\partial t^2}-c_1I_6\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial y^2}\right)+\\
&\quad+c_1\left[I_3\frac{\partial^2}{\partial t^2}\left(\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial x}\right)+J_4\frac{\partial^2}{\partial t^2}\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial \phi_y}{\partial x}\right)\right]\\
&\textbf{d)}\frac{\partial \overline{M}_{xx}}{\partial x}+\frac{\partial \overline{M}_{xy}}{\partial y}-\overline{Q}_x=\frac{\partial^2}{\partial t^2}\left(J_1u_0+K_2\phi_x-c_1J_4\frac{\partial w_0}{\partial x}\right);\quad \textbf{e)}\frac{\partial \overline{M}_{xy}}{\partial x}+\frac{\partial \overline{M}_{yy}}{\partial y}-\overline{Q}_y=\frac{\partial^2}{\partial t^2}\left(J_1v_0+K_2\phi_y-c_1J_4\frac{\partial w_0}{\partial y}\right)
\end{aligned}
\tag{5}
$$

where q is the external distributed load and with

$$
c_1=\frac{4}{3h^2},\quad c_2=\frac{4}{h^2}=3c_1;\quad \overline{M}_{\alpha\beta}=M_{\alpha\beta}-c_1P_{\alpha\beta};\quad \overline{Q}_\alpha=Q_\alpha-c_2R_\alpha \tag{6}
$$

$$
I_i=\sum_{k=1}^{N}\int_k^{k+1}\rho^{(k)}(z)^i dz,\ (i=0,1,2,\ldots 6);\ J_i=I_i-c_1I_{i+2},\ (i=1,4);\ K_2=I_2-2c_1I_4+c_1^2I_6 \tag{7}
$$

where α, β take the symbols x,y. The resultants (N_{xx}, N_{yy}, N_{xy}) denote the in-plane force resultants, M_{xx}, M_{yy}, M_{xy} the moment resultants, Q_x, Q_y the shear resultants and P_{xx}, P_{yy}, P_{xy} and R_x, R_y denote the higher-order stress resultants,

$$
\begin{Bmatrix} N_{\alpha\beta}\\ M_{\alpha\beta}\\ P_{\alpha\beta}\end{Bmatrix}=\int_{-h/2}^{h/2}\sigma_{\alpha\beta}\begin{Bmatrix}1\\ z\\ z^3\end{Bmatrix}dz;\begin{Bmatrix} Q_\alpha\\ R_\alpha\end{Bmatrix}=\int_{-h/2}^{h/2}\sigma_{\alpha z}\begin{Bmatrix}1\\ z^2\end{Bmatrix}dz \tag{8}
$$

4 RESULTS

We compute results for a simply supported FG plate comprised of aluminum and zirconia mainly because analytical results for a plate made of these materials is available for comparison[2]. Material properties of the aluminum (Al) and zirconia (ZrO_2) are: $\text{Al}: E_m=70\,GPa, \nu_m=0.3, \rho_m=2702\,kg/m^3$ $\quad \text{ZrO}_2: E_m=200\,GPa, \nu_m=0.3, \rho_m=5700\,kg/m^3$ We assume that the volume fraction of the ceramic phase is given by $V_c=V_c^-+\left(V_c^+-V_c^+\right)\left(1/2+z/h\right)^p$ where V_c^+ and V_c^- are, respectively, the volume fractions of the ceramic phase on the top and the bottom surfaces of the plate, and the parameter p dictates the volume fraction profile through the thickness. The estimation of the effective elastic constants follows the Mori-Tanaka technique[3] which was also used for the analytical solutions of Vel and Batra[2], and the meshless solutions of Qian et al.[4] and Ferreira et al.[1]. We consider multiquadric functions of the form $\sqrt{r+c}$ where r is the Euclidian distance between two nodes and c is a positive constante, taken here as $6d$ being d the distance between two consecutive nodes. In this paper we used regular grids, with equally spaced nodes.

p	*h/a*=0.05			*h/a*=0.10			*h/a*=0.20		
	Present	Ref. [4]	Exact	Present	Ref. [4]	Exact	Present	Ref. [4]	Exact
1	0.0147	0.0149	0.0153	0.0592	0.0584	0.0596	0.2188	0.2152	0.2192
2							0.2188	0.2153	0.2197
3							0.2202	0.2172	0.2211
5							0.2215	0.2194	0.2225

Table 1 Fundamental frequency of a simply supported square thick Al/ZrO2 FG plate, Mori-Tanaka scheme, third-order deformation theory, $V_c^-=0$, $V_c^+=1$

ceramic			*p*=1			metal		
N=7	N=11	Ref [4]	N=7	N=11	Ref [4]	N=7	N=11	Ref[4]
0.2468	0.2457	0.2469	0.2192	0.2188	0.2152	0.2121	0.2111	0.2122
0.4459	0.4483	0.4535	0.4047	0.3990	0.4114	0.3831	0.3852	0.3897
0.4462	0.4484	0.4535	0.4050	0.3992	0.4114	0.3834	0.3853	0.3897
0.5409	0.5395	0.5441	0.4818	0.4779	0.4761	0.4647	0.4636	0.4675
0.5410	0.5395	0.5441	0.4818	0.4779	0.4761	0.4648	0.4636	0.4675

Table 2 Five first natural frequencies of a simply supported square thick Al/ZrO2 FG plate, Mori-Tanaka scheme, third-order deformation theory, *h/a*=0.2, $V_c^-=0$, $V_c^+=1$

5 CONCLUSIONS

The collocation multiquadric radial basis function method was applied to the analysis of free vibrations of simply supported functionally graded plates. For the example tested, the present method shows excellent agreement with exact natural frequencies, particularly for thicker plates.

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