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AN OPTIMAL INTEGRATION SCHEME FOR THE VON-MISES CONSTITUTIVE MODEL BASED ON EXPONENTIAL MAPS

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Summary. *This paper focuses on a new integration scheme for the von-Mises elastoplastic consitutive model. Based on a time continuous re-formulation of the original model a proper integration scheme which makes use of an integration factor and of exponential maps is introduced. A comparison with previous and well established algorithms, in terms of iso-error maps, shows the main optimality characteristics of the new method.*

1 INTRODUCTION

The present study follows recent works[1,3], regarding a family of exponential-based integration algorithms for von-Mises associative plasticity with linear kinematic and isotropic hardening. The key point of the innovative numerical scheme presented in this work is the re-formulation of the original plasticity model as a quasi-linear dynamical system which can be approximated using exponential maps. The resulting exponential-based algorithm, differently from the one previously introduced[2], is proven to be consistent with the yield surface condition, exact in case of proportional loading and of zero isotropic hardening as well as second-order accurate.

The main aim of the present paper is then to compare the new algorithm with more classical methods such as return maps based on backward and generalized midpoint integration schemes.

2 TIME CONTINUOUS MODEL

We consider an *associative* von-Mises elastoplastic constitutive model with linear kinematic and isotropic hardening in the small deformation framework[4]. Following a standard deviatoric/volumetric splitting of the stress tensor $\boldsymbol{\sigma} = \mathbf{s} + p\mathbf{1}$ and strain tensor $\boldsymbol{\varepsilon} = \mathbf{e} + (1/3)\theta\mathbf{1}$ the model equations are

$$p = K\theta \tag{1}$$

$$\mathbf{s} = 2G\left[\mathbf{e} - \mathbf{e}^p\right] \tag{2}$$

$$\boldsymbol{\Sigma} = \mathbf{s} - \boldsymbol{\alpha} \tag{3}$$

$$F = \|\boldsymbol{\Sigma}\| - \sigma_y \tag{4}$$

$$\dot{\mathbf{e}}^p = \dot{\gamma}\mathbf{n} \tag{5}$$

$$\sigma_y = \sigma_{y,0} + H_{iso}\gamma \tag{6}$$

$$\dot{\boldsymbol{\alpha}} = H_{kin}\dot{\mathbf{e}}^p \tag{7}$$

$$\dot{\gamma} \geq 0, \quad F \leq 0, \quad \dot{\gamma}F = 0 \tag{8}$$

where K is the material bulk modulus, G is the shear modulus, $\mathbf{e}^p$ is the traceless plastic strain, Σ is the *relative stress*, α is the backstress, F is the von Mises yield function, $\mathbf{n}$ is the normal to the yield surface, σ_y is the yield surface radius, $\sigma_{y,0}$ the initial yield stress, H_{kin} and H_{iso} are respectively the kinematic and isotropic hardening moduli. Finally, Equations (8) represents the well known Kuhn-Tucker conditions. In what follows it is assumed that when $\dot{\gamma} = 0$ the system is in an elastic phase while, when $\dot{\gamma} > 0$ the system is in a plastic phase.

3 RE-FORMULATION AND INTEGRATION ALGORITHM

The stated problem can be put into a different form by combining Equations (2) and (3), deriving with respect to time and subsequently introducing Equations (5) and (7) respectively. Such calculations lead to the definition of the following integration factor

$$X_0(\gamma) = \begin{cases} \left(1 + \gamma H_{iso}/\sigma_{y,0}\right)^{\frac{2G + H_{iso} + H_{kin}}{H_{iso}}} & \text{if} \quad H_{iso} \neq 0 \\ \exp\left(\dfrac{2G + H_{iso} + H_{kin}}{H_{iso}}\gamma\right) & \text{if} \quad H_{iso} = 0 \end{cases} \tag{9}$$

which actually describes the evolution of the yield surface σ_y. Defining a generalized stress vector by means of

$$\mathbf{X} = \begin{pmatrix} X_0 \boldsymbol{\Sigma}/\sigma_y \\ X_0 \end{pmatrix} = \begin{pmatrix} \mathbf{X}^s \\ X_0 \end{pmatrix} \tag{10}$$

it is possible to reformulate the initial dynamical system in terms of the time derivative of vector (10). The form of this evolution law is the following

$$\dot{\mathbf{X}} = \mathsf{A}\mathbf{X} \tag{11}$$

with the matrix A depending on the **X** vector and on the actual phase

$$\mathsf{A} = \frac{2G}{\sigma_y}\begin{pmatrix} \mathbf{0} & \dot{\mathbf{e}} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad \text{elastic phase} \tag{12}$$

$$\mathsf{A} = \frac{2G}{\sigma_y}\begin{pmatrix} \mathbf{0} & \dot{\mathbf{e}} \\ \dot{\mathbf{e}}^{\mathrm{T}} & \mathbf{0} \end{pmatrix} \qquad \text{plastic phase} \tag{13}$$

The solution of Equation (11) can be approximated using exponential maps in each numerical time step. A detailed description of the new optimal exponential-based algorithm (in the sequel labelled as **ESC²**) and the full derivation of system (11) can be found in Reference[3].

4 NUMERICAL TESTS

In Figure 1 and Figure 2 we present iso-error maps[4] assuming a uniaxial elastic path up to yielding as starting point. We compare the following four algorithms:

MPT : the generalized midpoint method[5]
ESC² : the new exponential-based method derived from formulation (11)
ESC : the previously introduced non consistent exponential-based method[2]
RM : the well established and performing radial return map method[4]

The material properties adopted are E = 7000 N/m^2, $\nu = 0.3$, $\sigma_{y,0} = 24.3$ N/m^2, $H_{iso} = 225$ N/m^2, $H_{kin} = 0$ N/m^2.

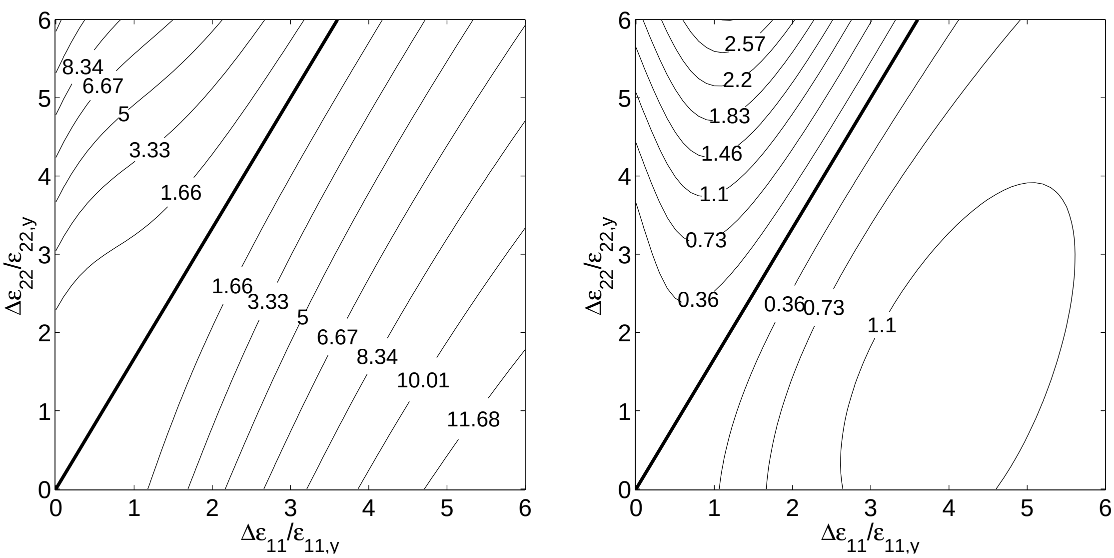

Figure 1: Iso-error maps: **MPT** and **ESC²** schemes.

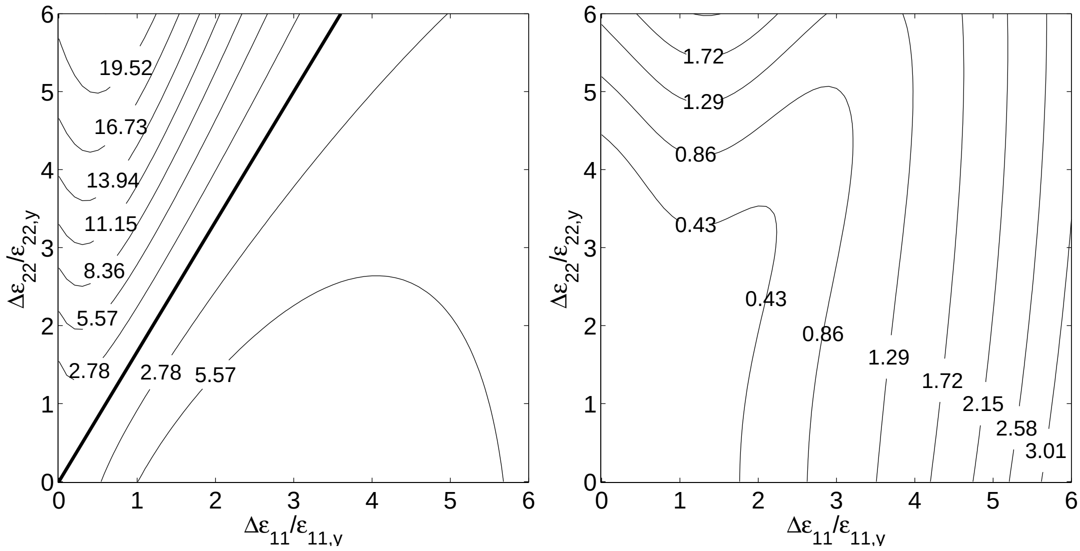

Figure 2: Iso-error maps: **RM** and **ESC** schemes.

It is assumed that each loading history is driven by controlling the increments of the ε_{11} and ε_{22} strain components while the remaining stresses are kept zero. It is evident that the error levels provided by the new **ESC**$^{\mathbf{2}}$ scheme are lower than the ones given by the **MPT** and the **RM** methods and that the new method presents zero error for proportional loading, while this does not hold true for the previous exponential-based **ESC** scheme.

5 CONCLUSIONS

We have presented a new exponential-based integration algorithm for von-Mises plasticity with linear hardening. The new scheme is competitive with more established methods and results exact in case of proportional loading and constant yield surface. Numerical tests show that the new procedure grants second order accuracy and a low error level even for large time step sizes.

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