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# ON THE EVALUATION OF THE ELASTOPLASTIC TANGENT STIFFNESS

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**Summary.** *A new method for the evaluation of the elastoplastic algorithmic tangent stiffness is presented. The formula is based on a simple geometric argument and does not involve matrix inversions. For regular yield surfaces we provide an explicit expression which is sufficiently approximate and computationally efficient and exact for homothetic yield surfaces such as in von Mises plasticity criterion.*

## 1 Introduction

The elastoplastic tangent stiffness is the linear operator which provides the stress rate corresponding to a prescribed strain rate. In computational mechanics, according to a fully implicit time integration scheme, the plastic flow rule is imposed in a time step by assuming that the plastic strain increment is normal to the final value of the stress state. The finite step flow rule is formulated by considering a purely elastic stress response to a given strain increment (the trial stress) and by performing the projection, in complementary energy, onto the convex elastic domain to get the right stress state. Accordingly the elastoplastic tangent stiffness must be reformulated as a consistent (algorithmic) tangent stiffness [1] which differs from the rate tangent stiffness if the trial stress is located outside the elastic locus and the yield surface is not flat. In geometric terms the algorithmic elastoplastic tangent stiffness is the composition between the derivative of the nonlinear projector on the convex elastic domain and the elastic stiffness. The evaluation of the derivative of the projector provides a new way to compute the algorithmic elastoplastic stiffness. A more important observation can be deduced by considering the hypersurface parallel to the boundary of the elastic domain and passing thru the trial stress point. The derivative of the nonlinear projector can then be expressed as the difference between the linear projector on the hyperplane tangent at the trial stress point and the shape operator of the parallel hypersurface thru the trial stress point times the distance between the trial stress and the projected stress. This simple expression provides a direct motivation why the algorithmic elastoplastic tangent stiffness is smaller then the elastoplastic tangent stiffness for nonflat yield surfaces (thus leading to a quadratic asymptotic convergence rate) and suggests a convenient way to avoid the matrix inversion operation usually involved in the computation of the algorithmic stiffness (see e.g. [2]). The trick consists in the substitution of the parallel hypersurface with the corresponding level set of the yield function (the one passing thru the trial stress point). This substitution provides the exact expression of the algorithmic stiffness when the level sets of the yield function form a family of homothetic surfaces (e.g. in von Mises plasticity criterion) and a simple useful approximation in the general case. Numerical examples have provided evidence of the advantages of the new estimate of the algorithmic stiffness.

## 2 Algorithmic tangent stiffness

Let us consider an elastoplastic problem characterized by a convex elastic domain $\mathcal{K}$ in the stress space $S$ and a finite step evolution problem in which the flow rule is imposed according to a fully implicit integration scheme. We denote by $S_{\mathbf{C}}$ and $D_{\mathbf{E}}$ the stress space and the strain space, respectively endowed with the inner products induced by the complementary elastic energy norm and the elastic energy norm. Let $\boldsymbol{\varepsilon}_0$, $\mathbf{p}_0$ and $\boldsymbol{\sigma}_0$ be the total, the plastic strain and total stress at the beginning of the step, and $\boldsymbol{\varepsilon}$, $\mathbf{p}$ and $\boldsymbol{\sigma}$ the corresponding strains and stresses at the end of the step. The constitutive equations are written as

$$\begin{cases} \mathbf{E}\,\boldsymbol{\varepsilon} = \boldsymbol{\sigma} + \mathbf{E}\,\mathbf{p}\,, \\ \mathbf{p} - \mathbf{p}_0 \in \mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma})\,, \end{cases}$$

where $\mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma})$ is the normal cone at $\boldsymbol{\sigma} \in \mathcal{K}$ in $S$.

An alternative form can be given in terms of the finite increments

$$\Delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0\,, \qquad \Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0\,, \qquad \Delta\boldsymbol{\sigma} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_0\,,$$

by setting

$$\begin{cases} \mathbf{E}\,\Delta\boldsymbol{\varepsilon} = \Delta\boldsymbol{\sigma} + \mathbf{E}\,\Delta\mathbf{p}\,, \\ \Delta\mathbf{p} \in \mathcal{N}_{\mathcal{K}}(\boldsymbol{\sigma})\,, \end{cases} \quad \Longleftrightarrow \quad \begin{cases} \boldsymbol{\sigma}_0 + \mathbf{E}\,\Delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} + \mathbf{E}\,\Delta\mathbf{p}\,, \\ \mathbf{E}\,\Delta\mathbf{p} \in \mathcal{N}^{\mathbf{C}}_{\mathcal{K}}(\boldsymbol{\sigma})\,, \end{cases}$$

where $\mathcal{N}^{\mathbf{C}}_{\mathcal{K}}(\boldsymbol{\sigma})$ is the normal cone at $\boldsymbol{\sigma} \in \mathcal{K}$ in $S_{\mathbf{C}}$. Denoting by $\mathbf{P}_{\mathcal{K}}$ the orthogonal projector in $S_{\mathbf{C}}$ onto $\mathcal{K}$ the constitutive relation may be written as

$$\boldsymbol{\sigma}_0 + \mathbf{E}\,\Delta\boldsymbol{\varepsilon} = \boldsymbol{\sigma} + \mathbf{E}\,\Delta\mathbf{p}\,, \qquad \boldsymbol{\sigma} = \mathbf{P}_{\mathcal{K}}\,(\boldsymbol{\sigma}_{\text{TR}})\,,$$

where $\boldsymbol{\sigma}_{\text{TR}} := \boldsymbol{\sigma}_{\mathbf{o}} + \mathbf{E}\,\Delta\boldsymbol{\varepsilon}$, and the finite step elasto-plastic constitutive problem may be formulated as:

$$\boldsymbol{\sigma} = \mathbf{P}_{\mathcal{K}}(\boldsymbol{\sigma}_{\text{TR}}) \quad \Longleftrightarrow \quad \begin{cases} \Delta\mathbf{p} = \lambda\, d\varphi(\boldsymbol{\sigma}) \\ \mathbf{C}\,\Delta\boldsymbol{\sigma} = \Delta\boldsymbol{\varepsilon} - \Delta\mathbf{p} \\ \lambda \geq 0 \quad \varphi(\boldsymbol{\sigma}) \leq 0 \quad \lambda\,\varphi(\boldsymbol{\sigma}) = 0\,. \end{cases}$$

Under plastic loading, i.e. when $\varphi(\boldsymbol{\sigma}) = 0$ we may write

$$\boldsymbol{\sigma} = \mathbf{P}_{\mathcal{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}) \quad \Longleftrightarrow \quad \begin{cases} \mathbf{C}\,\boldsymbol{\sigma} + \lambda\, d\varphi(\boldsymbol{\sigma}) = \mathbf{C}\,\boldsymbol{\sigma}_{\mathrm{TR}}\,, \quad \lambda \geq 0 \\ \varphi(\boldsymbol{\sigma}) = 0\,. \end{cases}$$

The classical procedure for the evaluation of the algorithmic tangent stiffness consists in taking the time derivatives of the previous relation to get $\dot{\mathbf{p}} = \dot{\lambda}\, d\varphi(\boldsymbol{\sigma}) + \lambda\, d^2\varphi(\boldsymbol{\sigma})\,\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\varepsilon}} - \mathbf{C}\,\dot{\boldsymbol{\sigma}}$. Setting $\mathbf{H} := [\mathbf{C} + \lambda\, d^2\varphi(\boldsymbol{\sigma})]^{-1}$ and imposing that the stress point moves along the boundary of the elastic domain we find the expression of plastic multiplier rate:

$$\dot{\boldsymbol{\sigma}} \in \mathcal{T}_{\partial\mathcal{K}}(\boldsymbol{\sigma}) \quad \Longleftrightarrow \quad \langle \dot{\boldsymbol{\sigma}}, d\varphi(\boldsymbol{\sigma}) \rangle = 0 \quad \Longrightarrow \quad \dot{\lambda} = \frac{\langle \mathbf{H}\,\dot{\boldsymbol{\varepsilon}}, d\varphi(\boldsymbol{\sigma}) \rangle}{\langle \mathbf{H}\, d\varphi(\boldsymbol{\sigma}), d\varphi(\boldsymbol{\sigma}) \rangle}\,.$$

Adopting the notations $\mathbf{N}_{\mathbf{H}} := \mathbf{H}\, d\varphi(\boldsymbol{\sigma})$ and $\beta := \langle \mathbf{H}\, d\varphi(\boldsymbol{\sigma}), d\varphi(\boldsymbol{\sigma}) \rangle$ the algorithmic tangent stiffness is then given by

$$\dot{\boldsymbol{\sigma}} = \big(\mathbf{H} - \frac{\mathbf{N}_{\mathbf{H}} \otimes \mathbf{N}_{\mathbf{H}}}{\beta}\big)\,\dot{\boldsymbol{\varepsilon}}$$

This expression requires a matrix inversion for the evaluation of $\mathbf{H}$. We provide here an alternative expression, which avoids matrix inversions, by a direct computation of the derivative of the projector operator, according to the formula

$$\dot{\boldsymbol{\sigma}} = \partial\mathbf{P}_{\mathcal{K}}\left(\boldsymbol{\sigma}_{\mathrm{TR}}\right) \mathbf{E}\,\dot{\boldsymbol{\varepsilon}}\,.$$

To this end we write $\boldsymbol{\sigma}_{\mathrm{TR}} = \mathbf{P}_{\mathcal{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}) + r\,\mathbf{n}(\boldsymbol{\sigma}_{\mathrm{TR}})$ where $r := \|\Delta\mathbf{p}\,\|$ and $\mathbf{n}(\boldsymbol{\sigma}_{\mathrm{TR}})$ is the outward normal at the point $\boldsymbol{\sigma}_{\mathrm{TR}} \in \mathcal{K}^r$. The expanded elastic domain $\mathcal{K}^r$ is obtained by moving in the outward direction the boundary $\partial\mathcal{K}$ of the convex elastic domain $\mathcal{K}$ along the vector field $r\,\mathbf{n}$. Differentiating the previous formula along a tangent vector $\mathbf{h} \in \mathcal{T}_{\partial\mathcal{K}^r}(\boldsymbol{\sigma}_{\mathrm{TR}})$ we get

$$\mathbf{h} = \partial_{\mathbf{h}}\mathbf{P}_{\mathcal{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}) + r\,\mathbf{S}(\boldsymbol{\sigma}_{\mathrm{TR}})\,\mathbf{h} \qquad \forall\,\mathbf{h} \in \mathcal{T}_{\partial\mathcal{K}^r}(\boldsymbol{\sigma}_{\mathrm{TR}})\,,$$

where $\mathbf{S}(\boldsymbol{\sigma}_{\mathrm{TR}}) := d\mathbf{n}(\boldsymbol{\sigma}_{\mathrm{TR}})$ is the shape operator of $\partial\mathcal{K}^r$ at the point $\boldsymbol{\sigma}_{\mathrm{TR}}$. Denoting by $\boldsymbol{\Pi}$ the linear orthogonal projector in $S_{\mathbf{E}}$ onto $\mathcal{T}_{\partial\mathcal{K}^r}(\boldsymbol{\sigma}_{\mathrm{TR}})$, the derivative of the projector $\mathbf{P}_{\mathcal{K}}$ may be written as

$$\partial\mathbf{P}_{\mathcal{K}}(\boldsymbol{\sigma}_{\mathrm{TR}}) = \boldsymbol{\Pi}(\boldsymbol{\sigma}_{\mathrm{TR}}) - r\,\mathbf{S}(\boldsymbol{\sigma}_{\mathrm{TR}})\,.$$

The shape operator $\mathbf{S}$ is a symmetric and positive by virtue of the convexity of the domain $\mathcal{K}^r$. We provide hereafter an approximate explicit expression of the shape operator $\mathbf{S}$ for an yield locus described by a single regular yield surface. The trick consists in the substitution of the parallel hypersurface $\partial\mathcal{K}^r$ with the corresponding level set of the yield function (the one passing thru the trial stress point). This substitution provides the exact expression of the algorithmic stiffness when the level sets of the yield function form a family of homothetic surfaces (e.g. in von Mises plasticity criterion) and a simple useful approximation in the general case. Denoting by $\|\cdot\|_{\mathbf{C}}$ the norm in $S_{\mathbf{C}}$, we consider the case when the convex elastic domain is characterized as the zero level set of a convex potential $\varphi \in \mathrm{C}^k(S_{\mathbf{C}}\,;\mathfrak{R})$, i.e.

$$\mathcal{K}^r := \{\, \boldsymbol{\sigma}_{\mathrm{TR}} \in S_{\mathbf{C}} \mid \varphi(\boldsymbol{\sigma}_{\mathrm{TR}}) \leq 0 \,\},$$

with boundary given by $\partial\mathcal{K}^r = \{\, \boldsymbol{\sigma}_{\mathrm{TR}} \in S_{\mathbf{C}} \mid \varphi(\boldsymbol{\sigma}_{\mathrm{TR}}) = 0 \,\}$. The gradient $\nabla\varphi \in \mathrm{C}^{(k-1)}(S_{\mathbf{C}}\,;S_{\mathbf{C}})$ of $\varphi$ in the space $S_{\mathbf{C}}$ is defined by

$$\langle\, d\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}), \dot{\boldsymbol{\sigma}}_{\mathrm{TR}} \,\rangle = \langle\, \mathbf{C}\, \nabla\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}), \dot{\boldsymbol{\sigma}}_{\mathrm{TR}} \,\rangle \quad \Longleftrightarrow \quad \nabla\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}) = \mathbf{E}\, d\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}),$$

$$\begin{aligned} \langle\, d^2\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\dot{\boldsymbol{\tau}}, \dot{\boldsymbol{\sigma}}_{\mathrm{TR}} \,\rangle &= \langle\, d(\mathbf{C}\, \nabla\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}))\dot{\boldsymbol{\tau}}, \dot{\boldsymbol{\sigma}}_{\mathrm{TR}} \,\rangle = \langle\, \mathbf{C}\, d(\nabla\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}))\dot{\boldsymbol{\tau}}, \dot{\boldsymbol{\sigma}}_{\mathrm{TR}} \,\rangle \\ &= \langle\, \mathbf{C}\, \nabla^2\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\dot{\boldsymbol{\tau}}, \dot{\boldsymbol{\sigma}}_{\mathrm{TR}} \,\rangle \quad \Longleftrightarrow \quad \nabla^2\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}) = \mathbf{E}\, d^2\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}), \end{aligned}$$

where $d\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}) \in D = L(S\,;\mathfrak{R})$ and $\langle \cdot, \cdot \rangle$ is the duality between $D$ and $S$. The outward normal versor at $\boldsymbol{\sigma}_{\mathrm{TR}} \in \partial\mathcal{K}$ is then given by

$$\mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}) = \frac{\nabla\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})}{\|\nabla\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\|_{\mathbf{C}}} \in S_{\mathbf{C}}\,.$$

The directional derivative of the normal versor $\mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}})$ along directions $\boldsymbol{\Pi}(\boldsymbol{\sigma}_{\mathrm{TR}})\,\dot{\boldsymbol{\sigma}}_{\mathrm{TR}}$ tangent to $\mathcal{K}$ at $\boldsymbol{\sigma}_{\mathrm{TR}}$ can be evaluated by a simple computation:

$$\begin{aligned} d\mathbf{n}_{\varphi} \cdot \boldsymbol{\Pi}\, \dot{\boldsymbol{\sigma}}_{\mathrm{TR}} &= \left( \frac{\nabla^2\varphi \cdot \boldsymbol{\Pi}\, \dot{\boldsymbol{\sigma}}_{\mathrm{TR}}}{\|\nabla\varphi\|_{\mathbf{C}}} - \frac{\langle\, \nabla\varphi, \nabla^2\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}) \cdot \boldsymbol{\Pi}\, \dot{\boldsymbol{\sigma}}_{\mathrm{TR}} \,\rangle}{\|\nabla\varphi\|^3_{\mathbf{C}}}\, \nabla\varphi \right) \\ &= \frac{1}{\|\nabla\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\|_{\mathbf{C}}} \Big( \mathbf{I} - \mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}) \otimes\, \mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}) \Big)\, \nabla^2\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}) \cdot \boldsymbol{\Pi}(\boldsymbol{\sigma}_{\mathrm{TR}})\, \dot{\boldsymbol{\sigma}}_{\mathrm{TR}} \\ &= \frac{1}{\|\nabla\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\|_{\mathbf{C}}}\, \boldsymbol{\Pi}(\boldsymbol{\sigma}_{\mathrm{TR}})\, \nabla^2\varphi(\boldsymbol{\sigma}_{\mathrm{TR}}) \boldsymbol{\Pi}(\boldsymbol{\sigma}_{\mathrm{TR}})\, \dot{\boldsymbol{\sigma}}_{\mathrm{TR}}\,. \end{aligned}$$

The approximate expression of the shape operator $\mathbf{S}$ of the surface $\partial\mathcal{K}^r$ is then given by

$$\mathbf{S}(\boldsymbol{\sigma}_{\mathrm{TR}}) = d\mathbf{n}_{\varphi}(\boldsymbol{\sigma}_{\mathrm{TR}}) \cdot \boldsymbol{\Pi}(\boldsymbol{\sigma}_{\mathrm{TR}}) = \frac{1}{\|\nabla\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\|_{\mathbf{C}}}\, \boldsymbol{\Pi}(\boldsymbol{\sigma}_{\mathrm{TR}})\, \nabla^2\varphi(\boldsymbol{\sigma}_{\mathrm{TR}})\, \boldsymbol{\Pi}(\boldsymbol{\sigma}_{\mathrm{TR}})\,.$$

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