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A NOVEL FORMULATION OF UPPER BOUND LIMIT ANALYSIS AS A SECOND-ORDER CONE PROGRAMMING PROBLEM

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Summary. *Here we present a dual kinematic formulation of limit analysis as a second-order cone programming problem, employing linear strain finite elements with a continuous displacement field. The result is a powerful tool for obtaining rigorous and tight upper bounds for very large discretized structures.*

1 INTRODUCTION

In this paper we describe an efficient approach to upper bound limit analysis of cohesive-frictional continua, using a finite element discretization in conjunction with second-order cone programming (SOCP). We also introduce the use of linear strain elements, focusing here on 6-node triangles in plane strain, though a directly analogous extension to 10-node tetrahedra in 3D is straightforward. If the vertices of such elements are taken as the flow rule points, it can be proved[1] that the solutions obtained are strict upper bounds on the exact collapse load. Two numerical examples using unstructured meshes show that the linear strain elements give better results than constant strain elements combined with kinematically admissible discontinuities (until now considered to be the only practical choice for a rigorous upper bound analysis using finite elements). The examples also demonstrate that, as in shakedown analysis[2], the use of SOCP is highly advantageous, allowing very large problems to be solved in 1-2 minutes on a desktop PC. The obvious limitation of this approach is that it can only be applied to quadratic cone-shaped yield functions (e.g. Mohr-Coulomb in plane strain, and Drucker-Prager).

2 FORMULATION OF OPTIMIZATION PROBLEM

A standard SOCP problem has the form

$$\begin{aligned} \min \quad & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x}_i \in \mathcal{C}_i \quad \forall i \in \{1,\dots,N\} \\ & \mathbf{x}^T = \begin{bmatrix}\mathbf{x}_1^T & \dots & \mathbf{x}_N^T\end{bmatrix} \end{aligned} \tag{1}$$

where $\mathbf{A} \in \Re^{m \times n}$, $\mathbf{b} \in \Re^m$, $\mathbf{c}, \mathbf{x} \in \Re^n$ and the sets $\mathcal{C}_i$ are second-order (or quadratic) cones of the form $\{\mathbf{x} \in \Re^d : \|\mathbf{x}_{2:d}\| \leq x_1,\ x_1 \geq 0\}$. For convenience we will employ the notation $(z, \mathbf{x}) \in \mathcal{C}$ as shorthand for $\|\mathbf{x}\| \leq z,\ z \geq 0$.

Consider a plane strain structure made of rigid–perfectly plastic material obeying the Mohr-Coulomb yield criterion (cohesion c, friction angle ϕ). The structure is discretized into 6-node triangular finite elements with straight sides. For these elements it can be shown that if the (associated) flow rule is enforced at the three vertices, it will automatically be satisfied throughout the whole element[1]. Upon applying the kinematic theorem, the arising SOCP optimization problem in the dual form reads

$$\begin{aligned} \max \quad & \beta \\ \text{s.t.} \quad & \sum_{i=1}^{NP} (A_i^e/3)\mathbf{B}_{\mathrm{m},i}\sigma_{\mathrm{m},i} + \sum_{i=1}^{NP} (A_i^e/3)\mathbf{B}_{\mathrm{d},i}\mathbf{s}_i^{\mathrm{red}} - \beta\mathbf{q} = \mathbf{q}_0 \\ & y_i + \sigma_{\mathrm{m},i}\sin\phi = c\cos\phi \quad \forall i \in \{1, \ldots, NP\} \\ & (y_i, \mathbf{s}_i^{\mathrm{red}}) \in \mathcal{C}_i \qquad\qquad\quad \forall i \in \{1, \ldots, NP\} \end{aligned} \tag{2}$$

where $\sigma_{\mathrm{m},i}$ and $\mathbf{s}_i^{\mathrm{red}} = \begin{bmatrix} s_{xx,i} & s_{xy,i} \end{bmatrix}^T$ are the mean and deviatoric stresses at the ith flow rule point, A_i^e is the area of the element to which the ith flow rule point belongs, $\mathbf{q}$ and $\mathbf{q}_0$ are load vectors, and the y_i are auxiliary variables. The matrices $\mathbf{B}_{\mathrm{m},i}$ and $\mathbf{B}_{\mathrm{d},i}$ incorporate the mean and deviatoric strain–displacement relations, according to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \mathbf{B}_{\mathrm{m},i}\mathbf{u} \quad \text{and} \quad \begin{bmatrix} \dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y} & \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{bmatrix}^T = \mathbf{B}_{\mathrm{d},i}\mathbf{u} \quad \text{at } (x, y) = (x_i, y_i)$$

3 NUMERICAL EXAMPLES

The analyses below were performed on a Dell PC (2.66 GHz CPU, 2 GB RAM) in the Windows XP environment, using the interior-point SOCP algorithm implemented in the MOSEK software package[3]. The meshes were produced using GiD, and in each case were used to compare the new 6-node linear strain triangles with the more usual configuration of 3-node constant strain triangles separated by discontinuities. It is noteworthy that, for a given mesh, the optimization problems obtained using the two different element types are of similar size, and thus the CPU times are broadly comparable as well. The quoted CPU times do not include the time spent reordering variables and rows during presolve (up to 20 s and 35 s for runs with the 6-node and 3-node elements respectively).

3.1 Rigid strip footing

The first example concerns the bearing capacity of a symmetrically loaded, rigid strip footing on purely frictional soil, in the absence of surcharge. The single mesh employed was unstructured – though very fine close to the footing edge – and consisted of 31481 elements. Analyses were performed for friction angles ranging from 10° to 40°. Table 1 gives the results, expressed in terms of the usual dimensionless factor $N_\gamma = 2q_{\mathrm{u}}/\gamma B$, and compares them with the exact values (coincident lower and upper bounds) that can be obtained using other numerical methods[4]. With the 6-node elements, the exact collapse load is overestimated by a maximum of 1.00% for a smooth footing and 2.62% for a rough footing (both when $\phi = 40°$). The upper bounds for all four friction angles are tighter than those of Hjiaj et al.[5], who employed 3-node elements, but with meshes that appear to have been specially tailored to provide favourably oriented discontinuities. If no such efforts are made to structure the mesh, Table 1 shows that the 6-node elements provide significantly better results, particularly for larger values of ϕ.

	6-node elements				3-node elements with discontinuities			
	smooth		rough		smooth		rough	
ϕ (°)	N_γ (error%)	CPU(s) (iter)	N_γ (error%)	CPU(s) (iter)	N_γ (error%)	CPU(s) (iter)	N_γ (error%)	CPU(s) (iter)
10	0.2820 (0.41)	77 (22)	0.4399 (1.55)	82 (24)	0.2853 (1.56)	66 (29)	0.4470 (3.20)	55 (26)
20	1.586 (0.44)	79 (23)	2.872 (1.16)	81 (24)	1.617 (2.40)	60 (28)	2.955 (4.09)	56 (27)
30	7.700 (0.61)	78 (23)	14.96 (1.37)	77 (23)	8.160 (6.63)	62 (29)	15.98 (8.34)	69 (33)
40	43.62 (1.00)	81 (24)	87.81 (2.62)	94 (26)	48.22 (11.66)	75 (35)	104.1 (21.70)	86 (41)

Table 1: Upper bounds on N_γ for various friction angles, for smooth and rough footings

3.2 Plane strain block with asymmetric holes

For the second example, three different meshes (814, 4446, 19714 elements) were generated for the structure shown in Figure 1. This problem has previously been analysed by Zouain et al.[6] using mixed elements (for $c = 1$ and $\phi = 0$), and by the authors[7] using lower bound stress elements (for $c = 1$ and $\phi = 0, 30°$). The results of the present analyses are given in Table 2, and they confirm the advantage of using the 6-node elements: much tighter upper bounds at a cost that is only slightly greater. In the absence of an exact solution, the 'error' entries in Table 2 have been calculated as $\frac{p^U - p^L}{p^U + p^L}$ where $p^L = 1.809c$ and $p^L = 1.056c$ are strict lower bounds for $\phi = 0$ and $\phi = 30°$ respectively[7]. Bracketing of the exact collapse load is very satisfactory: $\pm 0.44\%$ for $\phi = 0$ and $\pm 0.30\%$ for $\phi = 30°$.

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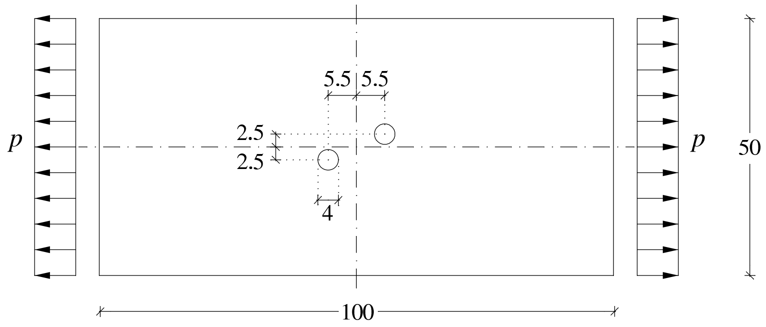

Figure 1: Plane strain block with asymmetric holes (after Zouain et al.[6])

	6-node elements				3-node elements with discontinuities			
	$\phi = 0$		$\phi = 30°$		$\phi = 0$		$\phi = 30°$	
NE	p/c (error%)	CPU(s) (iter)	p/c (error%)	CPU(s) (iter)	p/c (error%)	CPU(s) (iter)	p/c (error%)	CPU(s) (iter)
814	1.867 (1.59)	1.0 (16)	1.077 (0.96)	0.8 (16)	1.923 (3.06)	0.7 (20)	1.106 (2.29)	0.6 (18)
4446	1.845 (1.00)	7.3 (20)	1.070 (0.66)	5.3 (16)	1.896 (2.35)	5.0 (21)	1.104 (2.19)	4.7 (21)
19714	1.825 (0.44)	51.2 (17)	1.063 (0.30)	53.1 (24)	1.864 (1.49)	39.0 (21)	1.081 (1.16)	31.1 (22)

Table 2: Upper bound results for plane strain block

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