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INDIRECT ESTIMATION OF ESHELBY STRESS BY USE OF MIXED-TYPE FINITE ELEMENT

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Summary. *A numerical methodology of evaluating the Eshelby stress and its related equations is examined using a mixed-hybrid finite element method. The equilibrium of the Eshelby stress forms the canonical momentum in material space, and the Eshelby stress concept provides the theoretical foundation of fracture mechanics and electrodynamics. In order to clarify the role of the canonical momentum, a numerical algorithm is proposed based on the hybrid technique in which the strain and displacement are variants. The stress fields in notched plate and mode-I crack problem are implemented, and we discuss the differences from the Cauchy stress with respect to the stress distribution.*

1 INTRODUCTION

The Eshelby stress concept is applicable to a wide range of solid mechanics including cracks, inhomogeneity and electrodynamic effect[1]. Since the Eshelby stress is derived from the canonical transformation of the linear momentum in material space, the Eshelby stress field has certain physical meaning. How ever we encounter a difficulty in estimating the Eshelby stress due to the lack of boundary conditions in real space.

This paper deals with an indirect estimation of the Eshelby stress by use of mixed-hybrid finite element technique[2]. The strain and displacement are chosen for the variants in the hybrid formulation, by which C^0-class continuity is adopted also in the stress and strain. Stresses in notched root and crack tip are simulated through the technique, and we find a completely different aspect in the Eshelby stress from the ordinary Cauchy stress field.

2 BALANCE OF CANONICAL MOMENTUM

The balance of linear momentum in static state without body force read

$$\mathrm{Div}\,\boldsymbol{T} = 0 \quad , \tag{1}$$

in which $\boldsymbol{T}$ indicates the first Piola-Kirchhoff stress derived also from the energy density function $W(\boldsymbol{F})$ as

$$\boldsymbol{T} = \frac{\partial W}{\partial \boldsymbol{F}} \quad . \tag{2}$$

Here $\boldsymbol{F}$ denotes the deformation gradient. Applying the virtual work principle (or the weighted residual method) by use of the inverse motion $\delta\boldsymbol{x}=-\boldsymbol{F}\delta\boldsymbol{X}$, we have

$$\int \delta\boldsymbol{x}\cdot \mathrm{Div}\,\boldsymbol{T}dV = \int \delta\boldsymbol{X}\cdot\left(\mathrm{Div}\,\boldsymbol{b} + \boldsymbol{f}^{inh}\right)dV \quad . \tag{3}$$

Hence we obtain the following equations under material space:

$$\mathrm{Div}\,\boldsymbol{b} + \boldsymbol{f}^{inh} = 0 \quad , \tag{4}$$

$$\boldsymbol{b} = W\boldsymbol{1} - \boldsymbol{T}\boldsymbol{F} \quad , \tag{5}$$

$$\boldsymbol{f}^{inh} = -\mathrm{Grad}\,W\,|_{expl} \quad . \tag{6}$$

Equation (4) corresponds to the momentum balance equation (1) in material space. The second order tensor $\boldsymbol{b}$ is called “Eshelby stress” and $\boldsymbol{f}^{inh}$ means a kind of body force generating from the material inhomogeneity.

In order to solve these system of equations, boundary values have to be prescribed. However the surface force and prescribed displacement as well are not clear in material space, nor well-posed is the Eshelby stress. Hence we propose the ordinary solution procedure, then calculate the Eshelby stress based on Eq.(5), and examine the validity of Eq.(4). Then we are required to have the stress and strain with C^0-class continuity.

3 MIXED-HYBRID FORMULATION

Applying the Hellinger-Reissner principle, in which the stress and displacement are chosen for variants, we propose the following strain-displacement based variational principle:

$$\Pi\left(u_i,\varepsilon_{ij}\right) = \int_V E_{ijkl}\varepsilon_{kl}\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)dV - \int_V \frac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl}dV - \int_{S_t} u_i\bar{t}_i dS \quad . \tag{7}$$

Here we simply use the infinitesimal deformation theory and write down the equations in the Cartesian coordinate with the ordinary nomenclatures. Taking the first variation and equating to zero, we have basic equation for finite element method just corresponding to the virtual work principle. Since we employ the strain as a variant, we are able to obtain continuous strain and stress distribution in the objective domain.

In order to realize the finite element discretization, we use P1b-P1 element, in which a 4-node (bubble) triangular element is used for displacement and a 3-node linear triangular element for strain. Let the displacement be denoted by $\{u\}=[M]\{u^n\}$ and the strain by $\{\varepsilon\}=[N]\{\varepsilon^n\}$, the finite element stiffness can be divided into two equations as

$$\int[N]^T[E][\partial M]\{u^m\}dV - \int[N]^T[E][N]\{\varepsilon^n\}dV=\{0\} \ , \tag{8}$$

$$\int[\partial M]^T[E][N]\{\varepsilon^n\}dV - \int[M]^T\{\bar{t}\}dS=\{0\} \ . \tag{9}$$

The solution and the evaluation of the Eshelby stress are then straight forward.

4 NUMERICAL EVALUATION AND DISCUSSION

4.1 Stress field around notch root

The stress concentration in a square plate with a circular hole is firstly examined. A quarter part of the plate is examined. Figure 1 shows the stress distributions, in which the tensile stress (σ_{yy}) is demonstrated in the left side while the Eshelby stress (b_{yy}) is plotted in the right side. The balance of canonical momentum is checked and validated as it follows Eq.(4) under the absence of inhomogeneity term. As is well-known, the stress concentration appears around the notch root and the stress concentration factor takes about 3. However the Eshelby stress distribution is entirely different from the Cauchy stress, and inverse trend can be found around the notch root. It is impressive that even negative values are observed at the notch root. The second term of Eq.(5) may be dominant while the energy density is large at the notch root.

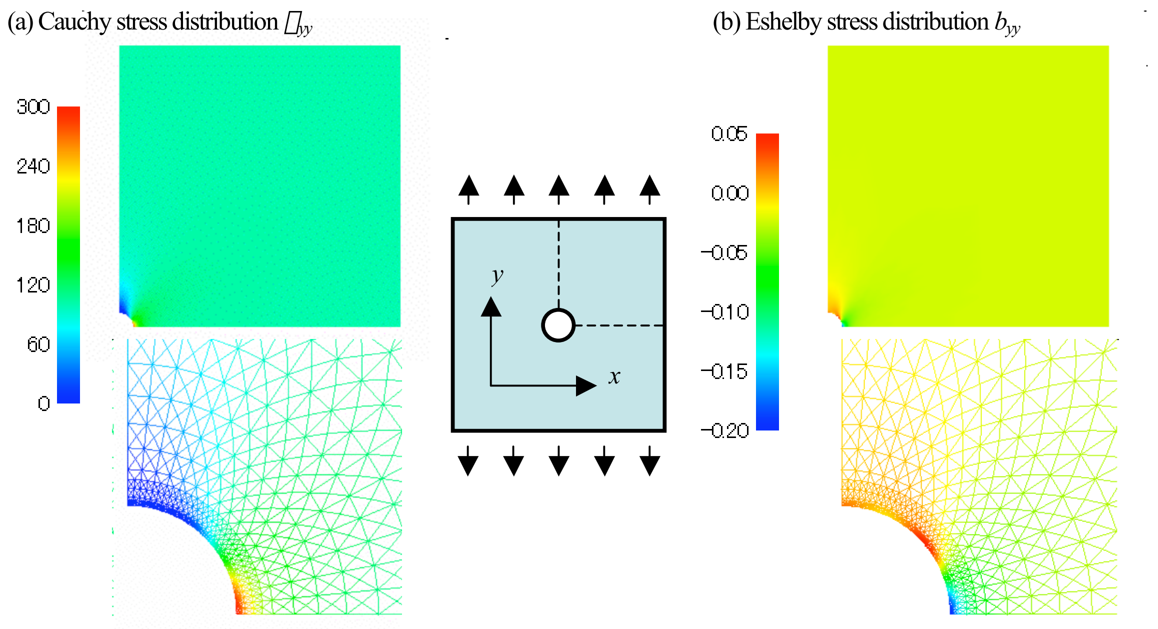

Fig.1: Stress concentration at notched root for a plate with a hole.

4.2 Mode-I crack

The stress pattern in mode-I cracking is shown in Fig.2. The stress at the crack tip is magnified in lower part. Here we also found the inverse tendency of these two stresses. Since the projection of the Eshelby stress along the crack implies the energy release, the negative sign may indicates the easiness of energy flow. This distribution would be a key point to grasp the physical meaning of the canonical momentum.

5 CONCLUSION

The mixed-hybrid finite element method is applied to the indirect evaluation of the Eshelby stress. The concept was proposed long time ago but is still unknown. A typical image of the stress distribution enhances the understanding of the profound insight of the canonical momentum theory.

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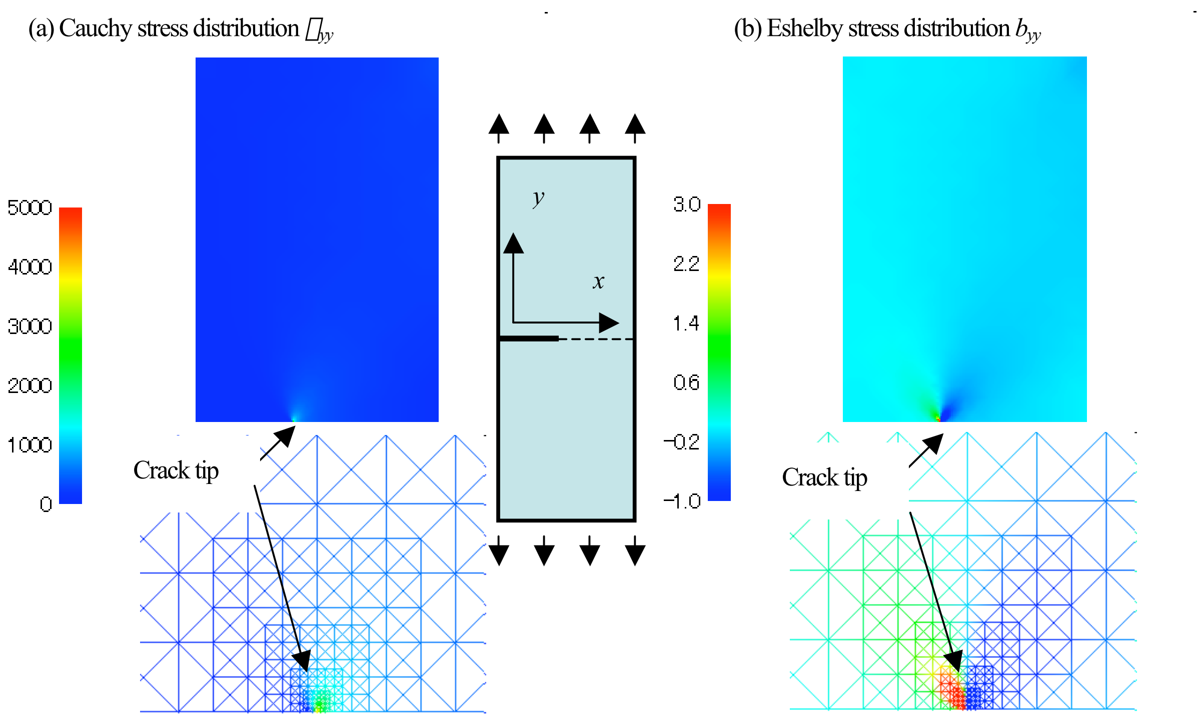

Fig.2: Stress concentration under mode-I cracking.