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CONFIGURATIONAL FORCES IN MULTIPLICATIVE ELASTO-PLASTICITY

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Summary. *In this contribution we aim at the elaboration of configurational forces in the context of multiplicative elasto-plasticity. The underlying decomposition of the deformation gradient is thereby considered as a representative and general kinematical framework for finite inelasticity.*

1 INTRODUCTION

Usual Newtonian mechanics address the movement of particles in physical space. Eshelbian or configurational mechanics, however, are essentially based on variations of the placement of particles in material space. Consequently the first approach is commonly denoted as the spatial motion problem while the second framework is referred to as the material motion problem. Configurational mechanics are of particular interest for the modelling of defects, inhomogeneities, heterogeneities and so forth in, e.g., solid mechanics; the main reason being that these phenomena are energetically conjugated to volume forces in configurational balance of linear momentum representations.

In this contribution we aim at the elaboration of material forces in the context of multiplicative elasto-plasticity, which is considered as a representative and general framework for finite inelasticity. The introduction of appropriate Eshelbian stress tensors and Eshelbian volume forces with respect to different configurations, namely the spatial, the material and - what we call - the intermediate setting, thereby turns out to be of cardinal importance.

Based on fundamental kinematic considerations, non-vanishing dislocation density tensors in terms of the plastic or elastic distortion can be introduced. These quantities, which apparently stem from the general incompatibility of the intermediate configuration, directly contribute to the intermediate volume forces. As a result, the obtained representations recapture the celebrated Peach-Koehler force which takes the interpretation of driving single dislocations. Contrary, material volume forces include, e.g., the gradient of the plastic distortion which implicitly incorporates dislocation density tensors.

2 ESSENTIAL KINEMATICS

Let the deformation gradient of the (sufficiently smooth) spatial motion problem, $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t)$ in $\mathcal{B}_t$, be decomposed via

$$\nabla_{\boldsymbol{X}} \boldsymbol{\varphi} = \boldsymbol{F} \doteq \boldsymbol{F}_{\mathrm{e}} \cdot \boldsymbol{F}_{\mathrm{p}} \quad \text{with} \quad J, J_{\mathrm{e}}, J_{\mathrm{p}} > 0 \tag{1}$$

wherein $J = \det(\boldsymbol{F})$, etc. The corresponding tangent map of the material motion problem, $\boldsymbol{X} = \boldsymbol{\Phi}(\boldsymbol{x}, t)$ in $\mathcal{B}_0$, consequently reads

$$\nabla_{\boldsymbol{x}} \boldsymbol{\Phi} = \boldsymbol{f} \doteq \boldsymbol{f}_{\mathrm{p}} \cdot \boldsymbol{f}_{\mathrm{e}} \quad \text{with} \quad j, j_{\mathrm{p}}, j_{\mathrm{e}} > 0 \tag{2}$$

wherein $j = \det(\boldsymbol{f})$, etc. In what follows we assume the combination of the spatial and material motion to render the identity mapping such that

$$\boldsymbol{f} \doteq \boldsymbol{F}^{-1}, \quad \boldsymbol{f}_{\mathrm{e}} \doteq \boldsymbol{F}_{\mathrm{e}}^{-1} \quad \text{and} \quad \boldsymbol{f}_{\mathrm{p}} \doteq \boldsymbol{F}_{\mathrm{p}}^{-1}, \tag{3}$$

respectively. For completeness, let appropriate velocity fields be denoted by

$$\boldsymbol{v} = \mathrm{D}_{\mathrm{t}} \boldsymbol{\varphi} \quad \text{with} \quad \boldsymbol{l} = \mathrm{d}_{\boldsymbol{x}} \boldsymbol{v} = \mathrm{D}_t \boldsymbol{F} \cdot \boldsymbol{f} \quad \text{and} \quad \boldsymbol{V} = \mathrm{d}_{\mathrm{t}} \boldsymbol{\Phi} \quad \text{with} \quad \boldsymbol{L} = \mathrm{D}_{\boldsymbol{X}} \boldsymbol{V} = \mathrm{d}_t \boldsymbol{f} \cdot \boldsymbol{F} \tag{4}$$

which results in the relation $\boldsymbol{v} = -\boldsymbol{F} \cdot \boldsymbol{V}$.

3 BALANCE OF LINEAR MOMENTUM

The classical format of balance of linear momentum is usually expressed in terms of, e.g., the spatial motion Piola stress tensor $\boldsymbol{\Pi}^{\mathrm{t}}$. When referring to both, the spatial as well as to the material motion problem (here for the static case) we end up with different representations which are related via Piola transformations, for instance

$$\nabla_{\boldsymbol{X}} \cdot \boldsymbol{\Pi}^{\mathrm{t}} + \boldsymbol{b}_0^{\mathrm{int}} + \boldsymbol{b}_0^{\mathrm{ext}} = \boldsymbol{0}, \quad \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\sigma}^{\mathrm{t}} + \boldsymbol{b}_t^{\mathrm{int}} + \boldsymbol{b}_t^{\mathrm{ext}} = \boldsymbol{0} \quad \text{with} \quad \boldsymbol{b}_t^{\mathrm{int,ext}} = j\, \boldsymbol{b}_0^{\mathrm{int,ext}} \tag{5}$$

$$\nabla_{\boldsymbol{X}} \cdot \boldsymbol{\Sigma}^{\mathrm{t}} + \boldsymbol{B}_0^{\mathrm{int}} + \boldsymbol{B}_0^{\mathrm{ext}} = \boldsymbol{0}, \quad \nabla_{\boldsymbol{x}} \cdot \boldsymbol{\pi}^{\mathrm{t}} + \boldsymbol{B}_t^{\mathrm{int}} + \boldsymbol{B}_t^{\mathrm{ext}} = \boldsymbol{0} \quad \text{with} \quad \boldsymbol{B}_0^{\mathrm{int,ext}} = J\, \boldsymbol{B}_t^{\mathrm{int,ext}} \tag{6}$$

wherein $\boldsymbol{b}_0^{\mathrm{int}} = \boldsymbol{0}$, $\boldsymbol{B}_0^{\mathrm{int}} \neq \boldsymbol{0}$ and further momentum representations being omitted.

4 COLEMAN-NOLL ENTROPY PRINCIPLE

With these relations in hand, let the strain energy density take the representation

$$J_{\mathrm{p}}\, W_{\mathrm{p}} = J\, W_t = W_0 = W_0(\boldsymbol{F}, \boldsymbol{F}_{\mathrm{p}}; \boldsymbol{X}) = W_0(\boldsymbol{F}_{\mathrm{e}}; \boldsymbol{X}) \tag{7}$$

so that the (isothermal) Dissipation inequality of the spatial motion problem results in

$$J_{\mathrm{p}}\, \mathsf{D}_{\mathrm{p}}^{\mathrm{loc}} = J\, \mathsf{D}_t^{\mathrm{loc}} = \mathsf{D}_0^{\mathrm{loc}} = \boldsymbol{\Pi}^{\mathrm{t}} : \mathrm{D}_t \boldsymbol{F} - \mathrm{D}_t W_0 - \boldsymbol{b}_0^{\mathrm{int}} \cdot \boldsymbol{v} = -\boldsymbol{\Pi}_{\mathrm{p}}^{\mathrm{t}} : \mathrm{D}_t \boldsymbol{F}_{\mathrm{p}} \geq 0 \tag{8}$$

whereby use of the abbreviations or rather hyperelastic formats

$$\boldsymbol{\Pi}^{\mathrm{t}} = \partial_{\boldsymbol{F}} W_0 |_{\boldsymbol{F}_{\mathrm{p}}} \quad \text{and} \quad \boldsymbol{\Pi}_{\mathrm{p}}^{\mathrm{t}} = \partial_{\boldsymbol{F}_{\mathrm{p}}} W_0 \big|_{\boldsymbol{F}} \tag{9}$$

has been made. By analogy, further Piola type stress tensors are introduced via

$$\boldsymbol{\pi}^{\rm t} = \partial_{\boldsymbol{f}} W_t|_{\boldsymbol{f}_{\rm p}}\,, \quad \boldsymbol{\pi}^{\rm t}_{\rm e} = \partial_{\boldsymbol{f}_{\rm e}} W_t\,, \quad \boldsymbol{\pi}^{\rm t}_{\rm p} = \partial_{\boldsymbol{f}_{\rm p}} W_{\rm p}\Big|_{\boldsymbol{f}} \quad \text{and} \quad \boldsymbol{\Pi}^{\rm t}_{\rm e} = \partial_{\boldsymbol{F}_{\rm e}} W_{\rm p}\,, \tag{10}$$

respectively. Correlated Cauchy (or rather Eshelby) type stresses consequently result in

$$\begin{aligned} \boldsymbol{\sigma}^{\rm t} &= W_t\,\boldsymbol{i}^{\rm t} - \boldsymbol{f}^{\rm t}\cdot\boldsymbol{\pi}^{\rm t} = W_t\,\boldsymbol{i}^{\rm t} - \boldsymbol{f}^{\rm t}_{\rm e}\cdot\boldsymbol{\pi}^{\rm t}_{\rm e}\,,\\ \boldsymbol{\Sigma}^{\rm t}_{\rm e} &= W_{\rm p}\,\boldsymbol{I}^{\rm t}_{\rm p} - \boldsymbol{F}^{\rm t}_{\rm e}\cdot\boldsymbol{\Pi}^{\rm t}_{\rm e} = W_{\rm p}\,\boldsymbol{I}^{\rm t}_{\rm p} - \boldsymbol{f}^{\rm t}_{\rm p}\cdot\boldsymbol{\pi}^{\rm t}_{\rm p}\,,\\ \boldsymbol{\Sigma}^{\rm t} &= W_0\,\boldsymbol{I}^{\rm t} - \boldsymbol{F}^{\rm t}\cdot\boldsymbol{\Pi}^{\rm t} = \boldsymbol{F}^{\rm t}_{\rm p}\cdot\boldsymbol{\Pi}^{\rm t}_{\rm p}\,. \end{aligned} \tag{11}$$

5 CONFIGURATIONAL VOLUME FORCES

In order to identify configurational volume forces we apply pullback transformations to the standard spatial motion balance of linear momentum representation. This approach recaptures on the one hand the hyperelastic formats highlighted above and, on the other hand, identifies the sought volume forces. In view of the material configuration we obtain, for instance, the relation

$$\boldsymbol{B}^{\rm int}_0 = -\,\boldsymbol{\Pi}^{\rm t}_{\rm p} : \nabla_{\boldsymbol{X}}\boldsymbol{F}_{\rm p} - \partial_{\boldsymbol{X}} W_0 \quad \text{and} \quad \boldsymbol{B}^{\rm ext}_0 = -\,J\,\boldsymbol{F}^{\rm t}\cdot\boldsymbol{b}^{\rm ext}_t\,. \tag{12}$$

Taking additionally the general incompatibility of the intermediate configuration into account, we furthermore observe

$$\boldsymbol{B}^{\rm int}_{\rm p} = -\,\boldsymbol{\Sigma}_{\rm e} : \boldsymbol{D}_{\rm e} - \boldsymbol{f}^{\rm t}_{\rm p}\cdot\partial_{\boldsymbol{X}} W_{\rm p} \quad \text{and} \quad \boldsymbol{B}^{\rm ext}_{\rm p} = -\,J_{\rm e}\,\boldsymbol{F}^{\rm t}_{\rm e}\cdot\boldsymbol{b}^{\rm ext}_t \tag{13}$$

which constitutes the configurational volume force related to the (intermediate) balance of linear momentum flux $\boldsymbol{\Sigma}^{\rm t}_{\rm e}$. Note that $\boldsymbol{D}^{\rm t}_{\rm e} = [\,\nabla^{\rm t}_{\boldsymbol{x}} \times \boldsymbol{f}_{\rm e}\,]\cdot{\rm cof}(\boldsymbol{F}_{\rm e})$ represents a dislocation density tensor with respect to the elastic distortion.

6 PEACH-KOEHLER FORCE

The intermediate configurational volume force is directly related to the celebrate Peach-Koehler force which drives the movement of a single single dislocation. In this context, we first introduce the correlated intermediate force

$$\mathbf{F} = \int_{\mathcal{V}_{\rm p}} \boldsymbol{B}^{\rm int}_{\rm p}\,{\rm d}V_{\rm p} = -\int_{\mathcal{V}_{\rm p}} \boldsymbol{\Sigma}_{\rm e}\times\boldsymbol{D}_{\rm e}\,{\rm d}V_{\rm p}\,, \tag{14}$$

whereby any dependence of the strain energy density on material placements $\boldsymbol{X}$ as well as external volume forces $\boldsymbol{b}^{\rm ext}_t$ have been neglected for conceptual clarity. Second, the incorporated dislocation density tensor is referred to solely one single dislocation, namely

$$\boldsymbol{D}^{\rm t}_{\rm e} = \delta_{\rm p}\,\boldsymbol{B}^{\rm bur}_{\rm p}\otimes\boldsymbol{T}_{\rm p} \tag{15}$$

wherein $\boldsymbol{B}^{\rm bur}_{\rm p}$ characterises the appropriate Burgers density, i.e. $\int_{\mathcal{A}_t}[\,\nabla^{\rm t}_{\boldsymbol{x}}\times\boldsymbol{f}_{\rm e}\,]\cdot\boldsymbol{n}\,{\rm d}A_t = \int_{\mathcal{A}_t} j_{\rm e}\,\boldsymbol{B}^{\rm bur}_{\rm p}\,{\rm d}A_t \neq \boldsymbol{0}$, and $\boldsymbol{T}_{\rm p}$ denotes the unit tangent vector according to the dislocation line. The Peach–Koehler force in the domain of interest consequently results in

$$\mathbf{F}^{\rm PK} = -\int_{\mathcal{V}_{\rm p}} \boldsymbol{\Sigma}_{\rm e}\times[\,\delta_{\rm p}\,\boldsymbol{B}^{\rm bur}_{\rm p}\otimes\boldsymbol{T}_{\rm p}\,]\,{\rm d}V_{\rm p} = -\int_{\mathcal{C}_{\rm p}} [\,\boldsymbol{\Sigma}_{\rm e}\cdot\boldsymbol{B}^{\rm bur}_{\rm p}\,]\times\bar{\boldsymbol{T}}\,{\rm d}S_{\rm p}\,. \tag{16}$$

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