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A $G\vartheta$-BASED METHOD FOR SIMULATING CRACK GROWTH IN DUCTILE MATERIALS

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Summary. *This work proposes a method for evaluating the increments of crack propagation in two-dimensional problems of fracture mechanics. A displacement-crack growth formulation based on the so-called ϑ-method is introduced for the case of rectilinear elastic crack growth. The formulation can be generalized for simulating crack propagation in ductile materials.*

1 INTRODUCTION

Nowadays the computer codes for simulation of crack growth in brittle and ductile materials (e.g. FRANC2D, FRANC3D, FRANC2D/L, ZENCRACK and, recently, ABAQUS) are still affected by a typical limit of the computational fracture mechanics: during crack propagation, the crack growth is not evaluated on the basis of a theoretical formulation. In fact the crack increments due to given loads are: (a) assigned after the determination of the direction of crack extension or (b) determined trough laws describing fatigue crack growth models (e.g. the empirical Paris law) [1].

In [2] Fortino and Bilotta proposed a coupled displacement-crack growth method for 2D linear elastic fracture mechanics which allowed the increments of displacement and crack growth relative to assigned load increments to be determined. The analysis used the classical equations of linear elasticity and a J integral-based criterion of crack propagation.

In this work a coupled displacement-crack growth formulation based on the $G\vartheta$ parameter [3] is introduced. As pointed out in [4] and [5], a $G\vartheta$-based approach is more suitable than the J integral theory in order to simulate general problems of curvilinear crack growth in ductile materials.

2 A $G\vartheta$-based displacement-crack growth formulation in elastic fracture

Let us analyze a two-dimensional elastic cracked body of domain $\Omega \in R^2$ and unit thickness with a rectilinear crack propagation (see Figure 1). No traction is applied along the surface of the crack. The material is assumed to be homogeneous and the body forces are neglected. The ϑ-method, originally introduced by Destuynder and Djaoua in [3], consists in a perturbation from the current configuration Ω to the updated configuration $\Omega_{\delta a}$ (see Figure 1):

$$\mathbf{F}_{\delta a}\mathbf{x} = \mathbf{x} + \delta a \boldsymbol{\vartheta}\,, \quad \forall \mathbf{x} \in \Omega \tag{1}$$

where $\mathbf{I}$ is the identity operator and $\boldsymbol{\vartheta}$ is a smooth vector valued function defined in Ω. In particular, $|\boldsymbol{\vartheta}| = 1$ at the crack tip and $\boldsymbol{\vartheta} = 0$ on $\partial\Omega \backslash S_f$.

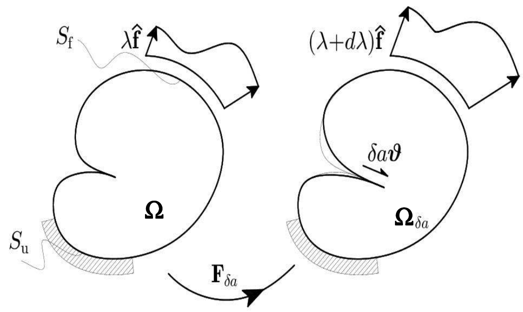

Figure 1: Current (Ω) and perturbed ($\Omega_{\delta a}$) cracked domains; S_f = boundary with applied forces; S_u = boundary with applied displacements; λ = control parameter; $\hat{\mathbf{f}}$ = fixed load; $\delta a > 0$ = crack length increment; $\delta\lambda$ = load increment.

The integral equations governing the elastic cracked problem in the updated configuration $\Omega_{\delta a}$ are rewritten in the reference configuration Ω in the following way:

$$\begin{cases} \int_\Omega \boldsymbol{\sigma}_{\delta a} : (\nabla \mathbf{v}\, \mathbf{J}_{\delta a}^{-1}) \mid \mathbf{J}_{\delta a} \mid \, d\Omega = (\lambda + \delta\lambda) \int_{S_f} \hat{\mathbf{f}} \cdot \mathbf{v}\, dS & \forall \mathbf{v} \in V \\ \int_\Omega \mathbf{C}\, \boldsymbol{\sigma}_{\delta a} : \boldsymbol{\tau} \mid \mathbf{J}_{\delta a} \mid \, d\Omega \, - \int_\Omega (\nabla \mathbf{u}_{\delta a}\, \mathbf{J}_{\delta a}^{-1}) : \boldsymbol{\tau} \mid \mathbf{J}_{\delta a} \mid \, d\Omega = 0 & \forall \boldsymbol{\tau} \in \Sigma \end{cases} \tag{2}$$

where $\mathbf{C}$ represents the compliance tensor, the spaces V and $\sum$ are

$$V = \{\mathbf{v} \in (H^1(\Omega))^2,\ \mathbf{v} = 0 \ \ on \ \ S_u\}; \ \ \Sigma = \{\boldsymbol{\tau} \in (L^2(\Omega))^4 | \ \boldsymbol{\tau}^T = \boldsymbol{\tau}\} \tag{3}$$

and $\mathbf{u}_{\delta a} = \mathbf{u} \circ \mathbf{F}_{\delta a}$; $\boldsymbol{\sigma}_{\delta a} = \boldsymbol{\sigma} \circ \mathbf{F}_{\delta a}$; $\mathbf{J}_{\delta a} = \nabla \mathbf{F}_{\delta a} = \mathbf{I} + \delta a (\nabla \vartheta)^T$.

By approximating the determinant of the Jacobian $\mathbf{J}_{\delta a}$ as $\mid \mathbf{J}_{\delta a} \mid \approx 1 + \delta a\, (div \boldsymbol{\vartheta})$ such that $(\mathbf{J}_{\delta a})^{-T} \approx 1 - \delta a (\nabla \boldsymbol{\vartheta})^T$ and substituting these expressions in (2), we find

$$\mathbf{u}_{\delta a} = \mathbf{u}_0 + \delta \mathbf{u} + R_u; \ \ \boldsymbol{\sigma}_{\delta a} = \boldsymbol{\sigma}_0 + \delta \boldsymbol{\sigma} + R_\sigma \tag{4}$$

where $(\mathbf{u}_0, \boldsymbol{\sigma}_0)$ is the solution in the reference configuration and $\delta a^{-1}(||R_u||_V + ||R_\sigma||_\Sigma) \rightarrow 0$ as $\delta a \rightarrow 0$. The increments $(\delta \mathbf{u}, \delta \boldsymbol{\sigma})$ are the unique solution in $V \times \Sigma$ of the following problem (see proof in [3]):

$$\begin{cases} \int_\Omega \delta\boldsymbol{\sigma} : \nabla\mathbf{v}\; d\Omega = \delta\lambda \int_{S_f} \hat{\mathbf{f}} \cdot \mathbf{v}\; dS + \delta a \int_\Omega \mathbf{s} : \nabla\mathbf{v}\; d\Omega & \forall \mathbf{v} \in V \\ \\ \int_\Omega \mathbf{C}\delta\boldsymbol{\sigma} : \boldsymbol{\tau}\; d\Omega - \int_\Omega \nabla\delta\mathbf{u} : \boldsymbol{\tau}\; d\Omega = \delta a \int_\Omega \mathbf{r} : \boldsymbol{\tau}\; d\Omega & \forall \boldsymbol{\tau} \in \Sigma \end{cases} \tag{5}$$

with $\mathbf{r} = -\frac{1}{2}(\nabla\mathbf{u}_0\nabla\boldsymbol{\vartheta} + (\nabla\mathbf{u}_0\nabla\boldsymbol{\vartheta})^T)$ and $\mathbf{s} = \boldsymbol{\sigma}_0\nabla\boldsymbol{\vartheta}^T - (div\boldsymbol{\vartheta})\boldsymbol{\sigma}_0$.

The new idea of this work is to use transformation (1) for writing an incremental displacement-crack propagation problem in the unknowns $(\delta\mathbf{u}, \delta a)$. At this end, eliminating the stresses we get $\delta\boldsymbol{\sigma} = \mathbf{E}[\boldsymbol{\varepsilon}(\delta\mathbf{u}) + \delta a\,\mathbf{r}]$, and (5) reduces to

$$\int_\Omega \mathbf{E}\boldsymbol{\varepsilon}(\delta\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\; d\Omega - \delta a \int_\Omega \mathbf{t} : \nabla\mathbf{v}\, d\Omega = \delta\lambda \int_{S_f} \hat{\mathbf{f}} \cdot \mathbf{v}\; dS \qquad \forall \mathbf{v} \in V;\;\; \delta a > 0 \tag{6}$$

where $\mathbf{t} = \mathbf{s} - \mathbf{E}\mathbf{r}$. In [3] the energy release rate of the cracked body during the perturbation from Ω to $\Omega_{\delta a}$ is proved to be equivalent to

$$G\vartheta = \frac{1}{2}\int_\Omega \boldsymbol{\sigma}_0 : \nabla\mathbf{u}_0(div\boldsymbol{\vartheta})\, d\Omega - \int_\Omega \boldsymbol{\sigma}_0 : (\nabla\boldsymbol{\vartheta})^T\nabla\mathbf{u}_0\, d\Omega \tag{7}$$

which coincides with the Rice J integral for all subdomains $\Omega_\vartheta \in \Omega$. In case of stable crack growth and flat resistance curves, the following conditions must be satisfied [1]:

$$G\vartheta = G_{\mathrm{f}};\;\; \delta G\vartheta = 0 \tag{8}$$

where G_{f} is the energy fracture and $\delta G_\vartheta = \int_\Omega \mathbf{t} : \nabla(\delta\mathbf{u})\, d\Omega - \delta a \int_\Omega \mathbf{E}\mathbf{r} : \mathbf{r}\; d\Omega$ represents the first variation of (7). Then, the coupled displacement-crack growth system in the unknowns $(\delta\mathbf{u}, \delta a)$ and will be

$$\begin{cases} \int_\Omega \mathbf{E}\boldsymbol{\varepsilon}(\delta\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\; d\Omega - \delta a \int_\Omega \mathbf{t} : \nabla\mathbf{v}\, d\Omega = \delta\lambda \int_{S_f} \hat{\mathbf{f}} \cdot \mathbf{v}\; dS \\ \\ -\int_\Omega \mathbf{t} : \nabla(\delta\mathbf{u})\, d\Omega + \delta a \int_\Omega \mathbf{E}\mathbf{r} : \mathbf{r}\; d\Omega = 0 \end{cases} \qquad \forall \mathbf{v} \in V;\;\; \delta a > 0 \tag{9}$$

and, in operator form:

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \delta\mathbf{u} \\ \delta a \end{bmatrix} = d\lambda \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix} \tag{10}$$

System (10) has been implemented and solved into the FEM computer code ELMER (CSC-Scientific Computing Ltd., Espoo, Finland) for problems of linear elastic fracture mechanics. Some results are shown in Figure (2).

3 Remarks and future work

The solution of system (10) inside a suitable algorithm including a remeshing or a non-remeshing technique, allows to determine the increments of displacement and crack growth relative to assigned load increments during rectilinear crack extension. The scheme

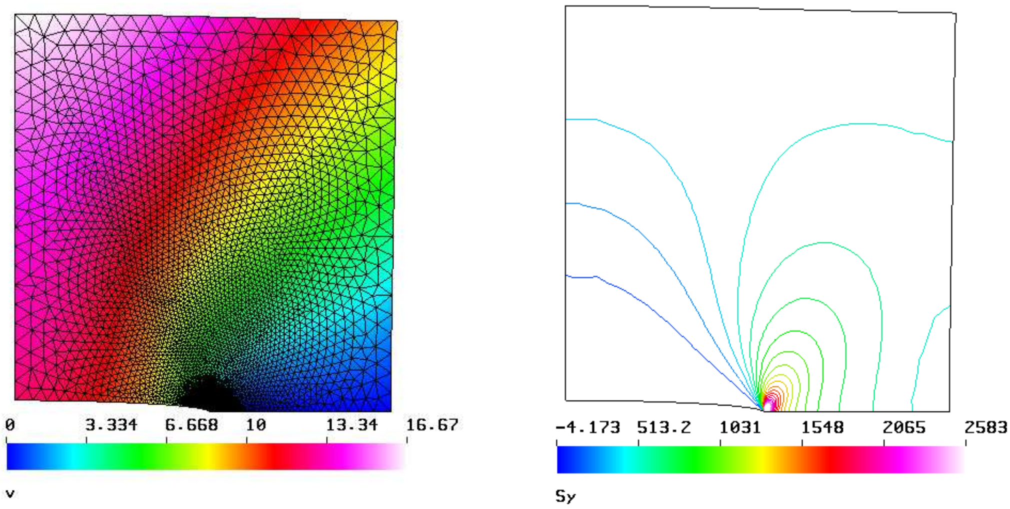

Figure 2: Center cracked plate in tension (one-quarter of geometry); dimensions = 20×20; E= 270; ν=0.3; initial crack length=10; $\lambda\hat{\mathbf{f}}$ = 100; SI units. Plane strain state. Quadratic triangular FE. Displacements and mesh (left), stresses σ_y (right) after the first step of a $G\vartheta$–based analysis; $\delta\lambda = 0.01$.

of the algorithm can be the same as that introduced in [2]. The method can be extended to elastic curvilinear crack growth by using the approach proposed by Rudoy in [4].

At present the authors are generalizing the proposed $G\vartheta$-based formulation in order to simulate crack propagation in ductile materials. In particular, to obtain the equivalent elastic-plastic of system (9), systems (2) and (5) have to be modified using the classical equations of incremental elastoplasticity. A first attempt to extend the ϑ-method to incremental elastoplasticity was proposed in [5] by Debruyne who introduced a fracture criterion but did not calculate the increments of crack growth during crack extension.

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