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ON THE VARIATIONAL FORMULATION OF NONLINEAR COUPLED THERMOMECHANICAL CONSTITUTIVE EQUATIONS AND UPDATES

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Summary. *A variational formulation of the coupled thermo-mechanical boundary-value problem for general dissipative solids is presented. The coupled thermo-mechanical boundary value problem under consideration consists of the equilibrium problem for a deformable, inelastic and dissipative solid with the heat conduction problem appended in addition. The variational formulation allows for general dissipative solids, including finite elastic and plastic deformations, non-Newtonian viscosity, rate sensitivity, arbitrary flow and hardening rules, as well as heat conduction.We showed that a joint potential function exists such that both the conservation of energy and the balance of linear momentum equations follow as Euler-Lagrange equations. This variational formulation of the thermo-mechanical boundary-value problem is then extended to the incremental framework.This allows to rewrite the update algorithm for the unknown fields over a given time (load) step as an optimization problem. The resulting variational update leads to an efficient numerical fully-coupled (i.e. monolithic) finite element formulation of thermo-mechanical problems.*

1 INTRODUCTION

This paper is concerned with the formulation of variational principles characterizing the solutions of the coupled thermo-mechanical problem for general dissipative solids, here understood as the equilibrium problem of an inelastic deformable solid to which the heat conduction problem is appended in addition. Problems of this nature arise in a variety of important fields of application, including: metal forming, machining, casting and other manufacturing processes; high-velocity impact such as ballistic penetration; and others. By a general dissipative solid we understand a deformable solid, possibly undergoing large deformations, possessing viscosity, internal processes and capable of conducting heat. By a variational characterization of the thermo-mechanical problem, we specifically mean the identification of a functional whose stationary points are solutions of the problem. Once this functional is known, the stable solutions of the problem may be identified with certain extrema of the functional, should any exist.

The ability to recast the coupled thermo-mechanical problem in variational form has a number of consequences and some beneficial effects. For instance, the variational framework opens the way for the application of the tools of calculus of variations to the analysis of the solutions of the problem. A variational statement of the problem also facilitates the formulation of numerical approximations, e.g., by means of Galerkin or Rayleigh-Ritz methods. In addition, in its time-discretized form the variational framework leads to the formulation of robust and efficient state-update algorithms in computational thermo-viscoplasticity [1].

2 VARIATIONAL FORM OF THE EQUATIONS

As shown in [2], the above thermomechanical equations can be restated under a variational form, generalizing the approach of Ortiz and Stainier [1], which was restricted to isothermal processes. Then, the solution to the thermomechanical boundary value problem in rate form derives from a variational principle:

$$\inf_{\dot{\boldsymbol{\varphi}},\boldsymbol{H},T,\dot{N},\dot{\boldsymbol{Z}}} \quad \Phi(\dot{\boldsymbol{\varphi}},\boldsymbol{H},T,\dot{N},\dot{\boldsymbol{Z}}) \ . \tag{1}$$

where Φ is a suitably chosen potential function of the deformation mapping $\boldsymbol{\varphi}$, the heat flux $\boldsymbol{H}$, the temperature T, the entropy N, and the collection of internal variables $\boldsymbol{Z}$. Taking the first variations in (1) yields the mechanical balance law, the conduction law, the heat equation in entropy form, and the kinetic equations for internal variables. It is noteworthy that, unlike the potential functions proposed by Biot [3, 4] and subsequently others [5, 6, 7], the potential Φ in (1) is directly built from general physical quantities, e.g., the internal energy, entropy and kinetic potential, without any assumptions peculiar to their specific forms.

3 VARIATIONAL CONSTITUTIVE UPDATES

A time-discretized version of the variational form outlined in the previous section can be established. Let us consider an incremental procedure, and more particularly a generic time interval $[t_n, t_{n+1}]$. Let the fields describing the state of the body at time t_n ($\boldsymbol{\varphi}_n$, $\boldsymbol{H}_n$, T_n, N_n, $\boldsymbol{Z}_n$) be known. We then seek to characterize the updated fields, at time t_{n+1}, as *optimizers* of a suitably chosen function Φ_n, inspired from the above potential Φ. The updated fields are then solutions to the following variational principle:

$$\inf_{\boldsymbol{\varphi}_{n+1},\boldsymbol{H}_{n+1},T_{n+1},N_{n+1},\boldsymbol{Z}_{n+1}} \quad \Phi_n(\boldsymbol{\varphi}_{n+1},\boldsymbol{H}_{n+1},T_{n+1},N_{n+1},\boldsymbol{Z}_{n+1};\boldsymbol{\varphi}_n,\boldsymbol{H}_n,T_n,N_n,\boldsymbol{Z}_n) \ . \tag{2}$$

4 CONVERSION OF PLASTIC WORK TO HEAT

The study on heating from plastic power may be traced back to the pioneering work of Taylor and Quinney, who performed their seminal experimental work in 1937 [8]. In

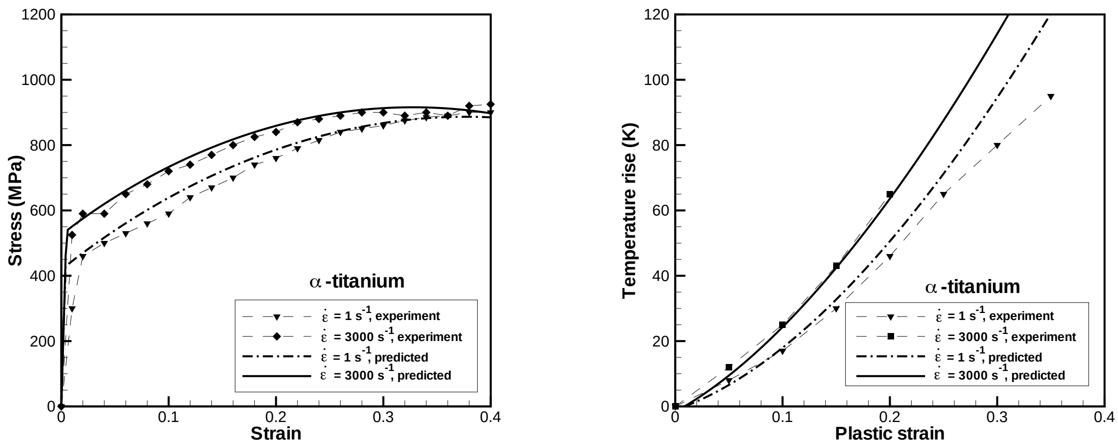

Figure 1: Stress-strain curves and adiabatic temperature rise as a function of strain for α-titanium at strain rates of 1 s^{-1} and 3000 s^{-1}.

previous theoretical and numerical analyses, it has often been assumed that the local rate of heating is a constant fraction of the plastic power. But, for engineering materials, experimental results show considerable variation of the heat to plastic-work ratio with both strain and strain rate. Within a variational framework the rate of conversion of plastic work into heat is an outcome of the theory and cannot be modelled independently. This prediction of the theory provides a test of whether thermo-mechanical behavior of a solid is variational. We performed this test for two different materials: α-titanium and Al2024-T3. In particular, we calculated temperature histories under the conditions of the Kolsky (split-Hopkinson) pressure bar experiment and compared the predictions with the measurements of Hodowany et al. [9]. These results are illustrated in Figures 1 and 2. The good correspondence between the theoretical predictions and the experimental data suggests that both α-titanium and Al2024-T3 are variational.

5 CONCLUSIONS

We have developed a variational framework for the coupled thermo-mechanical boundary-value problem for general dissipative solids. In this framework, the equations of motion and the energy balance equation follow jointly as Euler-Lagrange equations of a common potential function. The equilibrium constitutive relations and kinetic relations of the material also follow by taking variations with respect to the internal variables as in previous variational formulations of the isothermal case [1].

REFERENCES

[1] M. Ortiz and L. Stainier. The variational formulation of viscoplastic constitutive updates. *Comput. Meth. Appl. Mech. Eng.*, **171**, 419–444, 1999.

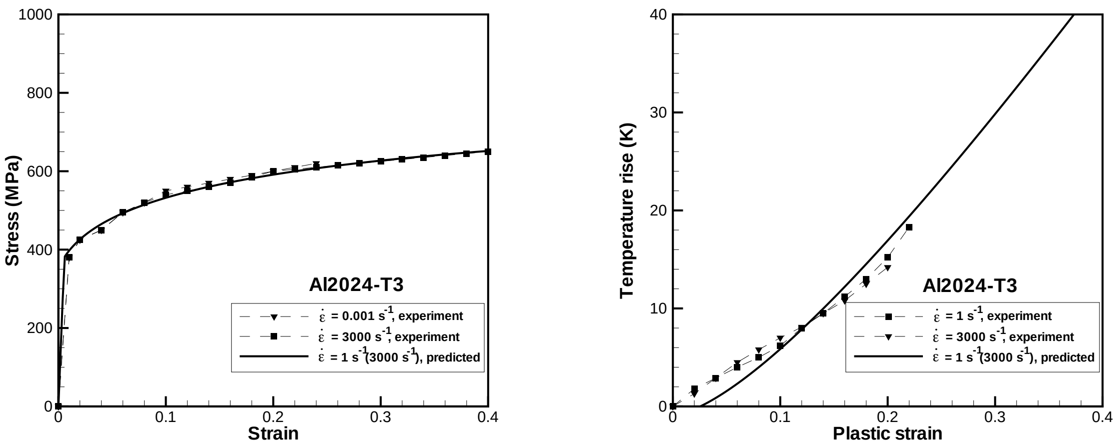

Figure 2: Stress-strain curves and adiabatic temperature rise as a function of strain for Al2024-T3 at strain rates of 1 s^{-1} and 3000 s^{-1}.

[2] Q. Yang, L. Stainier, and M. Ortiz. A variational formulation of the coupled thermo-mechanical boundary-value problem for general dissipative solids. *J. Mech. Phys. Solids*, 2005. Submitted for publication.

[3] M.A. Biot. Thermoelasticity and irreversible thermodynamics. *J. Stat. Phys.*, **27**, 250–253, 1956.

[4] M.A. Biot. Linear thermodynamics and the mechanics of solids. *Proc. Third U.S. National Congress of Applied Mechanics*, ASME, 1–18, 1958.

[5] M. Ben-Amoz. On a variational theorem in coupled thermoelasticity. *J. Appl. Mech.*, **32**, 943–945, 1965.

[6] G. Herrmann. On variational principles in thermoelasticity and heat conduction. *Quarterly Appl. Math.*, **22**, 151–155, 1963.

[7] G. Batra. On a principle of virtual work for thermo-elastic bodies. *J. Elasticity*, **21**, 131–146, 1989.

[8] G.I. Taylor and H. Quinney. The latent heat remaining in a metal after cold working. *Proc. Royal Soc. London*, **A163**, 157–181, 1937.

[9] J. Hodowany, G. Ravichandran, A.J. Rosakis, and P. Rosakis. Partition of plastic work into heat and stored energy in metals. *Exper. Mech.*, **40**, 113–123, 2000.