Improved Normals and Curvatures From 2D Volume Fractions

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Among numerous methods for numerical simulation of two-phase flows, e.g. wave impact to marine vessels and ocean structures, the volume-of-fluid (VoF) method is one that can exactly preserve the volume of each phase [4, 5]. In the VoF method, the volume fraction f of one phase is considered as a reference, and defined in each cell as the ratio of the volume occupied by that phase to the total cell volume. Cells with $f = 1$ and $f = 0$ are then full of one or the other fluid; those cells with $0 < f < 1$ define the interface between the phases. Such a volume fraction field is discontinuous, and so it is difficult to obtain accurate estimates of interface normals and curvatures, required both by the geometric advection algorithms that are typical of VoF methods [6], and for the surface tension term that must be calculated by an accompanying flow solver [2]. Here, we present two new methods for calculating normals and curvatures from volume fractions.

Many methods have been developed for computing normals [3, 4, 5]. One family (e.g. Youngs) calculates simple gradients of the volume fractions, but such methods yield normals that don't converge with refinement, and curvatures that in fact diverge with refinement. Other, more complicated, methods fit curves to the volume fraction data and then evaluate normals and curvatures from the curves, but such methods are complex and expensive because they obtain the curve fits iteratively to minimize some objective function (e.g. LVIRA, which yields first order accurate normals).

The height function (HF) method is gradient based, and has been used extensively due to its simplicity and accuracy [1]. In the HF method, predicting the interface orientation first with a simple technique, a stencil is created and fluid heights H are obtained:

$$H_{i+\alpha} = \sum_{\beta=-3}^{3} f_{i+\alpha,j+\beta}\Delta y_{j+\beta} \quad \text{for } \alpha = -1, 0, 1 \tag{1}$$

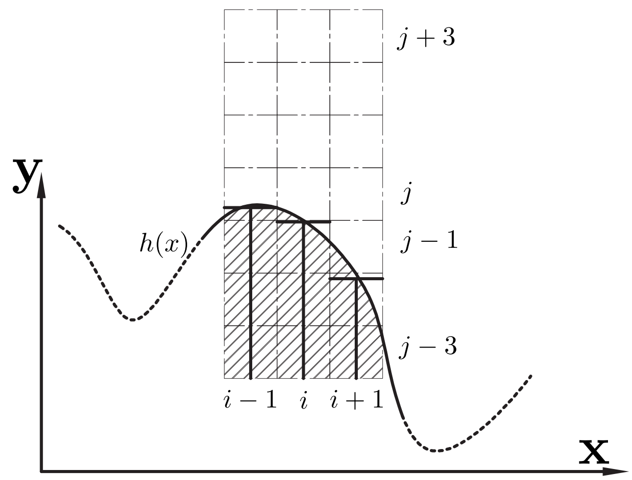

Figure 1: Height function definition.

where Δy is the vertical dimension of a cell (see Figure 1). The line normal (n_x, n_y) and curvature κ can then be calculated by the central difference method:

$$\begin{aligned} n_y &= H_{i+1} - H_{i-1} \\ n_x &= x_{i+1} - x_{i-1} \\ \kappa &= \frac{\ddot{y}}{(1 + (n_y/n_x)^2)^{\frac{3}{2}}} \end{aligned} \tag{2}$$

where $\ddot{y}$ can be evaluated by central difference approximations in terms of the heights. Here, we introduce an enhancement which lead to (i) fully second-order accurate normals and (ii) more accurate curvatures via a hybrid HF/curve fit method, that yields exact results for circular arcs and better normals and curvatures for arbitrary curves. An additional benefit of this method is more accurate normals and curvature even for underresolved volume fraction data.

A Hybrid HF / Arc Fit Least-square Technique: A more complex way to compute normals and curvature is to use a fitting curve like an arc, and combine the least-square and HF concepts. This method uses the concept of an osculating

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circle which is the circle that best approximates the curve at a point. The discretized error function is defined as follows and minimized iteratively subject to a constraint to obtain the best fit:

$$\tilde{E}_{i,j}^2 = \left[\left(\tilde{H}_{i-1} - H_{i-1}\right)^2 + \left(\tilde{H}_{i+1} - H_{i+1}\right)^2 \right]^{\frac{1}{2}} \tag{3a}$$

$$\tilde{f}_{i,j} = f_{i,j} \tag{3b}$$

where $\tilde{H}$ and $\tilde{f}$ are the heights and volume fractions obtained by the approximating arc, respectively. Since an arc is fixed by three values (its centre and radius), it is possible to eliminate the error and reduce these equations to a simple system which can be solved by a nonlinear root finding technique like Newton-Raphson. When the arc is obtained, the normal at the midpoint of the arc segment confined by the cell is considered the interface normal, and the reciprocal of the arc radius is the curvature of the interface. This method obtains normals and curvature for a line and a circle up to machine precision.

The new method introduced here calculates normals and curvature more accurately than any other method we are aware of, particularly in under-resolved regions.

References

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