Adjoint methods for finding periodic orbits

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ABSTRACT

When finding periodic orbits we wish to find a state of the system, $\underline{\mathrm{x}}(0)$, to which the system returns after time T. This is equivalent to minimizing the cost function $\mathcal{J} = ||\underline{\mathrm{x}}(T) - \underline{\mathrm{x}}(0)||^2$, by varying the starting system state $\underline{\mathrm{x}}(0)$ and the period T.

We use variational methods to derive a set of adjoint equations from the governing equations. By integrating the direct equations forward in time, and then the adjoint equations backward in time, the adjoint method finds the exact gradient of the cost function at the initial state. The gradient can then be used to minimise $\mathcal{J}$ in a steepest descent optimisation algorithm, or in other gradient methods, such as the conjugate-gradient algorithm. Similarly, an additional forward-backward time integration can yield second order differentials of the cost function (a Hessian-vector product), which can be used to increase the accuracy of line search routines. These second order derivatives do not suffer from the same inaccuracies that can occur from simple finite differences.

In the adjoint method, the gradient of the cost function is found in one forward-backward time integration, irrespective of the dimension of the system. This includes the gradient with respect to the period of the orbit and other important system parameters. This could result in faster optimisation routines for large systems, where the number of functional evaluations has a strong effect on total calculation time. The adjoint techniques described in the paper are demonstrated by finding bifurcations in a thermoacoustic system.

REFERENCES

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