Computing two- and three-dimensional cross fields
A PDE approach based the Gizburg-Landau theory

Jean-François Remacle†, Pierre-Alexandre Beaufort† , François Henrotte† and Jonathan Lambrechts †

†Université Catholique de Louvain, MEMA, Avenue Georges Lemaitre 4, bte L4.05.02, 1348 Louvain-la-Neuve, Belgium

ABSTRACT

Assume a 2D domain $\Omega \subset \mathbb{R}^2$ with its smooth boundary $\partial\Omega$. A frame field is a set of two orthonormal vector fields definied on Ω. More precisely, a frame f is a member of the quotient space $f \in \mathrm{S}^1/Q$ where S^1 is the circle group and Q is the group of quadrilateral symmetry. In the context of mesh generation, a frame field represent at each point of the domain the preferred orientations of a quadrilateral mesh. The frame fields that are considered for mesh generation should then be aligned with the boundary. The usual way of building smooth frame fields in 2D is to propagate frames from the boundary $\partial\Omega$ to the interior of the domain. Different approaches have been proposed for extending frames in the domain, some of them being based on solving some partial differential equations. In this paper, we propose an innovative approach to construct frame fields in 2 and 3 dimensions that is based on the Ginzburg-Landau theory [1]. In 2D, we have

$$\min_f E(f) = \underbrace{\frac{1}{2}\int_\Omega |\nabla f|^2 \, d\Omega}_{\text{Smoothing}} + \underbrace{\frac{1}{4\epsilon^2}\int_\Omega \left(1-|f|^2\right)^2 d\Omega}_{\text{Penality}}. \tag{1}$$

Energy (1) has two terms. The first part is a smoothing operator that forces frames to vary slowly and the second part is a penality term that vanishes on S^1/Q. The asymptotic behavior of $E(f)$ is known when ϵ is small: this has interresting consequenses that will be discussed.

For three-dimensional frame-fields, the GinzburgLandau problem (1) becomes

$$\min_f E(f) = \frac{1}{2}\int_\Omega |\nabla f|^2 \, d\Omega + \frac{1}{4\epsilon^2} W(f). \tag{2}$$

where the function f takes its value the 9-dimensional space of fourth order spherical harmonics in which the manifold SO(3)/O of the configurations of the cube is embedded isometrically and the function W is a suitable function that vanishes on SO(3)/O. The structure of the singularities in the 3D problem is not known. Yet, solutions can be computed numerically and the results are used to build high quality hex-dominant meshes [2].

REFERENCES

[1] Bethuel, F. and Brezis, H. *Ginzburg-Landau Vortices*. Springer Science and Business Media, 2012.

[2] Baudouin, T. C., Remacle, J. F., Marchandise, E., Henrotte, F., and Geuzaine, C., A frontal approach to hex-dominant mesh generation. *Advanced Modeling and Simulation in Engineering Sciences*, Vol. **1**, pp. 1–30, (2013).