On the best constant matrix approximating an oscillatory matrix in divergence-form operators

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Abstract

We address in this work the very simple, linear, elliptic, in conservative form problem

$$- \operatorname{div}(A_\varepsilon \boldsymbol{\nabla} u_\varepsilon) = f, \tag{1}$$

on a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \geq 2$), where A_ε is a highly oscillatory matrix, and f a loading term. We assume that the conditions are satisfied so that (1) is well-posed. In our mind, this problem represents an idealization of a practical situation where Ω is a material, which physical properties are encoded in the coefficient A_ε, of characteristic scale of variation ε, and u_ε is the response of this material to the loading f. From a numerical viewpoint, for ε close to zero, discretizing this problem beyond the characteristic scale is not affordable if the solution is needed for a large number of loadings. Hence, different approaches have been developed to efficiently and accurately compute the solution of (1), depending on the scale of variation ε of the material properties. For ε almost zero, homogenization theory is an answer [3], while for intermediate regimes with not too small characteristic scale, one can think for example of MsFEM-type methods [1]. However, there are several situations for which such approaches fail.

- In practical situations, the coefficient A_ε is often only partially known, or completely unavailable, and one only has access to the response u_ε of the material for some loadings (assuming that this response can be observed/measured at scale ε). In this case, homogenization theory or MsFEM-type approaches are unusable as they cruelly hinge on the knowledge of the real analytical oscillations of the material properties to encode them in the resulting average model or multiscale basis.

- Suppose now that one examines the homogenized limit of (1) for a coefficient A_ε that is not the simple rescaling $A(\cdot/\varepsilon)$ of a simple (periodic, quasi-periodic, random stationary, ...) function. Or, equivalently, that one tries to homogenize more complicated settings, or more complicated equations. Then it might very well be the case that, although A_ε is well-known, and although the homogenized limit of (1) is known to read $- \operatorname{div}(A^\star \boldsymbol{\nabla} u^\star) = f$, for some homogenized matrix $A^\star$, no explicit expression of $A^\star$ is available. Similar difficulties arise when the available expressions for $A^\star$ are not amenable to efficient numerical computations in practice (think, e.g., of random stationary cases with long-range correlations).

In the traces of [2], we investigate in this work an alternative pathway to accurately and efficiently approximate the solution of (1) in all situations, and especially in situations where standard techniques fail or are inefficient. Restricting ourselves to the case where problem (1) averages toward an homogenized equation with constant matrix $A^\star$, we exhibit, regardless of how 'large' the small parameter ε may be, the best constant matrix $\overline{A_\varepsilon}$ for which the solution of $- \operatorname{div}(\overline{A_\varepsilon} \boldsymbol{\nabla} \overline{u_\varepsilon}) = f$ (note that $\overline{u_\varepsilon}$ is non-oscillating) best approximates (in some sense) the one of (1). The matrix $\overline{A_\varepsilon}$ is obtained as the solution of an optimization problem, for a competitive computational workload, and obviously tends to $A^\star$ as ε tends to zero.

References

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