On divergence-free finite element methods for incompressible flows with vortical structures

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We consider Galerkin-FEM for the transient incompressible Navier–Stokes equations:

$$\begin{cases} \text{Find } (\boldsymbol{u}, p) : (0,T) \to \boldsymbol{\mathcal{V}} \times \mathcal{Q} \text{ s.t.} \\ \partial_t \boldsymbol{u} - \nu \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} + \nabla p = \boldsymbol{f}, \\ \qquad\qquad\qquad\qquad\quad \nabla \cdot \boldsymbol{u} = 0. \end{cases}$$

In particular, we apply conforming and inf-sup stable methods with $\nabla \cdot \boldsymbol{\mathcal{V}}_h \subseteq \mathcal{Q}_h$, thereby yielding exactly divergence-free approximations to the velocity field. This choice of interpolation spaces covers some popular isogeometric methods [1, 2].

After a brief overview concerning the conservation properties of the considered FEM [3], we provide velocity error estimates for the case $\boldsymbol{u} \in \mathcal{L}^\infty\left(0, T; \boldsymbol{\mathcal{W}}^{1,\infty}(\Omega)\right)$ which are semi-robust with respect to the Reynolds number and independent of the pressure approximation [4].

Concluding, applications to problems from vortex dynamics are considered where standard FEM are compared to divergence-free ones. We demonstrate the superiority of divergence-free methods for flows with vortical structures. Further, we show that convection stabilisation can improve the discrete solution drastically, both qualitatively and quantitatively.

KEY WORDS: Incompressible viscous flow, divergence-free FEM, inf-sup stability, pressure and semi-robust error estimates, vortex dynamics.

REFERENCES

1. A. Buffa, C. de Falco, and G. Sangalli. IsoGeometric analysis: Stable elements for the 2D Stokes equation. *Int. J. Numer. Meth. Fluids*, 65(11–12):1407–1422, 2011.
2. J. A. Evans and T. J. R. Hughes. Isogeometric divergence-conforming B-splines for the steady Navier–Stokes equations. *Math. Models Methods Appl. Sci.*, 23(8):1421–1478, 2013.
3. S. Charnyi, T. Heister, M. A. Olshanskii, and L. G. Rebholz. On conservation laws of Navier–Stokes Galerkin discretizations. Preprint #51, Department of Mathematics, University of Houston, 2016.
4. P. W. Schroeder and G. Lube. Pressure-robust analysis of divergence-free and conforming FEM for evolutionary incompressible Navier–Stokes flows. arXiv preprint `https://arxiv.org/abs/1610.03224`, 2016.