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ON THE USE OF THE RECIPROCITY GAP PRINCIPLE FOR SOLVING SOME INVERSE PROBLEMS IN HYDROGEOLOGY

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Summary. *We introduce in this work a new algebraic process for the identification of unknown parameters in groundwater problems [6]. This method is borrowed from the inverse problem mathematical community, it is based on the so-called Reciprocity Gap principle [1,2]. It allows the recovering of parameters from the knowledge of internal or boundary hydraulic heads.*

1 INTRODUCTION

The efficiency of aquifer modelling as a tool for the knowledge of the groundwater resources and for the design of sustainable groundwater management plans, depends obviously on the degree of our knowledge of the hydrogeological parameters of the aquifer in hand, such as transmissivity, storage, well fluxes, aquifer recharges *etc*. Determining unknown physical parameters by fitting the model to observed heads is the so-called **inverse problem**.

The trial-and-error procedure is the simplest way to solve inverse problems. More sophisticate methods consist on replacing this manual method by an optimisation one. From these remarks, Neuman [7] classified methods of estimating parameters as direct or indirect according to whether the parameters are obtained directly using the head distribution as known in the differential equation governing the flow, or whether they are obtained indirectly as a non-linear optimisation problem for which a set of calculated heads is matched to the observed one.

We introduce in this paper, a new algebraic process for the identification of unknown hydraulic parameters (for more details see [6]). The method, based on the so-called Reciprocity Gap principle (RG), has been introduced in [1,2] for the inverse geometrical problem of planar cracks identification. It has been widely exploited within the Inverse Problems mathematical community essentially for recovering unknown geometries from over-specified boundary data. It is inspired on the well-known Maxwell-Betti Reciprocity Principle.

To illustrate this method we will apply it for recovering transmissivities, storages, well's fluxes [5] and aquifer recharge in both transient and stationary cases.

2 MATHEMATICAL IDENTIFICATION PROCESS

2.1 The groundwater model

Combining the conservation equations and the Darcy's law we derive the following boundary value problem which describes flow in porous saturated media [3]:

$$\begin{cases} S(x,y)\dfrac{\partial h}{\partial t} + div(-T(x,y)\nabla(h)) = f & in \quad \Omega \\ h(x,y,t) = g & on \quad \Gamma_D \\ T\dfrac{\partial h}{\partial n} = \Phi & on \quad \Gamma_N \\ h(x,y,0) = H_0 & in \quad \Omega \end{cases} \tag{1}$$

Where *S(x,y)* is the storage function, *T(x,y)* the transmissivity function, *f* the source term which can include well's fluxes or/and surfaced recharging/evaporation, *g* the Dirichlet boundary condition which represents the prescribed heads on a part Γ_D of the domain boundary and Φ the prescribed flux on the remaining boundary Γ_N .

The forward problem, or the aquifer simulation, consists in calculating the piezometric level *h* from the knowledge of all the other parameters (*S, T, f, g* and *Φ*). On the other side, the inverse problem consists of recovering the parameters, assuming that the piezometric heads are known.

2.2 The Reciprocity Gap Principle

Our parameters identification process is an algebraic one. It is based on the reciprocity gap principle (RG). The RG is inspired from the well-known Maxwell-Betti reciprocity principle which is equivalent to the virtual works theorem.

Roughly speaking, RG compare the response of a defective body to that of the safe body having the same characteristics to a given solicitation. In the inverse problem dealing with recovering transmissivities or storages the 'safe' is a reference homogenous body. While in the case of well's fluxes or surfaced recharging\evaporation identification, the 'safe' is the domain with source term $f = 0$.

The main idea of the method is to multiply the first equation of the system (1) with harmonic fields *v* (*div(grad v) = 0*) then we integrate it over all the domain *Ω* and we use Green' second formula so that we obtain a linear system.

Let us apply this method for two examples:

Example n°1: We consider the case of recovering well's fluxes. We suppose that we are in a stationary situation with known transmissivity T and that the source term f has this expression:

$$f = \sum_k Q_k \, \delta_{P_k} \tag{2}$$

With Q_k is the well abstraction corresponding to a point source located at P_k with coordinates (x_k,y_k) and δ is the Dirac distribution.
Multiplying the first equation of (1) by v, integrating it over all the domain Ω and using Green' second formula we find:

$$R(v) = \sum_k Q_k v(P_k) \tag{3}$$

Where:

$$R(v) = \int_{\partial\Omega} T\left(\frac{\partial h}{\partial n} v - \frac{\partial v}{\partial n} h\right) \, d\Gamma \tag{4}$$

Then we exploit the reciprocity gap functional with various particular fields v.
Notice that the left hand side of the equality (3 or 4) is totally known and depends only on the boundary data.
More precisely, for k wells with unknown fluxes we evaluate $R(z^j)$ where z is the complex variable (the real and imaginary part of z^j are harmonic). Then equation (3) is written:

$$R(z^j) = \sum_k Q_k \, z_k^j \tag{5}$$

where $z_k = x_k + \mathrm{i}y_k$ is the affix of the point source P_k.
Therefore the determination of the fluxes of a collection of wells amounts to solving a linear system :

$$AQ = b \tag{6}$$

Where:

- $A = (z^i_j)$ is the Wandermonde matrix with z the complex variable,
- $Q=(Q_j)$ the unknown fluxes vector and
- $b=(R(z^i))$ the known right hand side vector of equation (4) in which $v=z^i$.

Note that in the case of recovering surface recharging\evaporation quantities, only equation (3) becomes:

$$R(v) \quad = \quad \sum_k E_k \int_{\Omega_k} v dS \tag{7}$$

Where E_k is the recharging\evaporation quantities and Ω_k the surface of recharge\evaporation. So we obtain a linear system to solve :

$$B E \quad = \quad b \tag{8}$$

Where:

- the general coefficient of the matrix B is : $B_{ij} = \int_{\Omega_j} z^i$,
- $E = (E_j)$ the unknown recharge vector and
- $b = (R(z^i))$ the same vector as the case of identifying wells' fluxes.

Example n°2: In the case of storage identification, we suppose that $S(x,y)$ is a piecewise constant unknown function ,T, f, g and Φ are known and that the piezometric head h is known at the initial time ($h = H_0(x,y)$) and the final time T_f ($h = H_f(x,y)$) in all the domain Ω. Appling the RG principle we find:

$$R(v) = \sum_k \int_{\Omega_k} S_k (H_f - H_0) v \quad d\Omega \tag{9}$$

With:

$$R(v) \quad = \int_{\partial\Omega} T v \left(\int_{T_0}^{T_f} \frac{\partial h}{\partial n} \quad dt \right) d\Gamma \quad - \int_{\partial\Omega} T \frac{\partial v}{\partial n} \left(\int_{T_0}^{T_f} h \quad dt \right) \quad d\Gamma + \int_{T_0}^{T_f} dt \int_{\Omega} f v \quad d\Omega \tag{10}$$

Therefore the identification of the storage coefficients amounts to solving a linear system:

$$C S \quad = \quad c \tag{11}$$

Where:

- the general term of matrix C is: $C_{ij} = \int_{\Omega_j} (H_f - H_0) v_i \quad d\Omega$,
- $S=(S_j)$ is the unknown storage coefficient vector and
- $c=(R(z^i))$ the right hand side vector of equation (9) in which $v=z^i$.

Let us point out that the accessible data in the inverse problem under consideration in example n°1 corresponds to a **complete boundary** data *h*. Whereas in the case of storage identification as well as transmissivity identification we need a **complete interior** data *h*. As, generally, in practice we have the value of the hydraulic head, only in few measurement points, we resort to interpolate methods like kriging to deduce the complete interior data.

3 NUMERICAL TRIALS

The observed values (free of measurements error) are first generated by solving a forward problem by finite element analysis, computations have been run on FemLab [4]. Then the inverse problem is solved by applying the reciprocity principle.

We consider two cases, one in a stationary situation and the other in the transient situation. For the steady case we deal with the recharge/evaporation identification, while in the non-steady study we recover the storage coefficients.

3.1 Stationary case

We consider the synthetical case of a heterogeneous domain with two surface recharging zones, as shown on figure 1. The studiedy domain is a rectangular 1000mx250m, with two vertical impermeable boundaries and two horizontal prescribed head boundaries. The domain is meshed with a regular mesh of triangular elements with linear interpolation, characterized by 3063 nodes and 1468 elements.
Note that as shown on equation (4), in this case, we need the knowledge of heads measurements only on the **boundaries of the domain**.

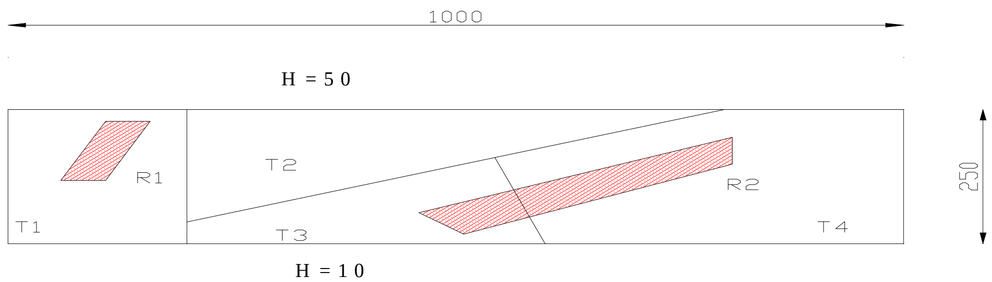

Figure 1: Heterogeneous case with two unknown recharged surface

Actual recharge values, computed ones and the relative error are shown on Table 1. We note that the identification is efficient since the relative error is acceptable and less than 15%.

$R_{i\ exact}$ ($m^3\ s^{-1}$)	2.5 10^{-3}	1 10^{-6}
$R_{i\ calc}$ ($m^3\ s^{-1}$)	2.2 10^{-3}	0.89 10^{-6}
Relative Error (%)	12	11

Table 1: Exact recharge values ($R_{i\ exact}$) and computed ones ($R_{i\ calc}$)

3.2 Transient case

a- Storage identification with complete interior data:

We consider the synthetical case of a heterogeneous domain with two storage coefficients, as shown on figure 2. The studied domain is a rectangular 1000m x 500m with transmissivity T = 20m^2/day and an uniform recharge varying from 5 10^{-4} m/day at the initial time to 3 10^{-4} m/day at 30 days. Boundary conditions are two vertical impermeable boundaries and two horizontal prescribed head boundaries (h_{upper} = 20m and h_{lower} = 10m). The domain is meshed with a regular mesh of triangular elements with linear interpolation, characterized by 981 nodes and 466 elements.

Note that as shown on equation (11), in this case, we need **complete interior** but only at the initial and final times

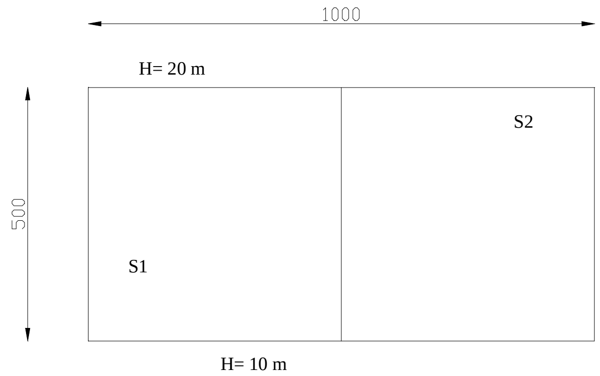

Figure 2: Studied case with two unknown storage coefficients

$S_{i\ exact}$ ($m^3\ s^{-1}$)	0.01	0.0001
$S_{i\ calc}$ ($m^3\ s^{-1}$)	0.0102	0.00011
Relative Errors (%)	2%	11%

Table 2: Exact storage coefficient ($S_{i\ exact}$) and computed ones ($S_{i\ calc}$)

b- Storage identification with interpolated data: With this last example we consider, in the storage identification procedure, piezometric data which are interpolated from local measurements in the domain instead of the hypothetic situation where these data are supposed to be available in the entire domain. For this, we use the values of the hydraulic heads on some points as shown on Fig 3 and interpolate them with the kriging method, using the software Surfer [8], before computing the integrals for equations (9) and (10).

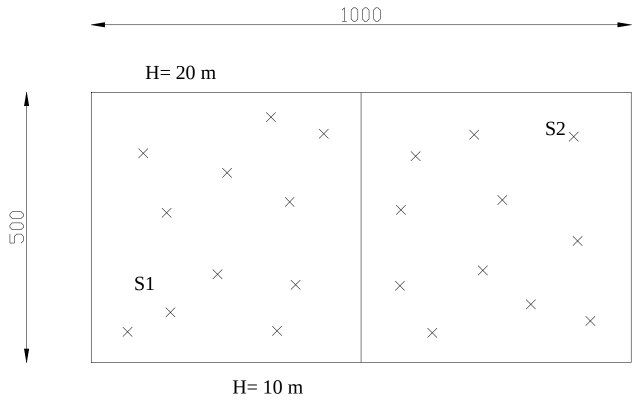

Figure 3: The case with two unknown storage coefficients with measurement points (x).

$S_{i\ exact}$ ($m^3\ s^{-1}$)	0.01	0.0001
$S_{i\ calc}$ ($m^3\ s^{-1}$)	0.011	0.00012
Relative Error (%)	12	21

Table 3: Exact storage coefficient ($S_{i\ exact}$) and computed ones ($S_{i\ calc}$) in the kriging case

4 CONCLUSIONS

- We present here a new explicit method, based on the Reciprocity Gap Principle, for the identification of aquifers parameters from known piezometric levels data.
- This identification process provides an algebraic set of equations very easy to implement.
- The numerical trials that we performed have shown the efficiency of the present method : in all cases, the error on the parameter recovered remains acceptable
- For the last example, this algebraic identification process was preceded by a kriging interpolation step to obtain the hydraulic head in the whole domain. Even in this case the numerical results remain satisfactory.
- The presented method is very cheap (both from the computational time and the implementation process), it can be exploited to give an appropriate first guess in an iterative parameters recovering method.

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