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NUMERICAL METHODS FOR WEAKLY NONLINEAR STABILITY ANALYSIS OF SHALLOW WATER FLOWS

Mohamed S. Ghidaoui* **and Andrei A. Kolyshkin†**

*Hong Kong University of Science and Technology
Clear Water Bay, Kowlooon, Hong Kong, China
e-mail: ghidaoui@ust.hk, web page: http://ihome.ust.hk/ ghidaoui/

†Riga Technical University
1 Meza street LV1007 Riga, Latvia
e-mail: akoliskins@rbs.lv

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Summary. Numerical methods for calculation of the coefficients of the Ginzburg-Landau equation describing the evolution of the most unstable mode in a shallow water flow within the framework of a weakly nonlinear theory are described in the present paper. Chebyshev collocation method is used to compute the eigenvalues and eigenfunctions of a linear stability problem. Approximation in the form of Chebyshev polynomials is also applied to solve three boundary value problems (one of which is resonantly forced) whose solutions are used later to compute the coefficients of the Ginzburg-Landau equation.

1 INTRODUCTION

Shallow water flows are widespread in nature and engineering. Examples include flows in compound and composite channels, jets, mixing layers and wakes. Large-scale two dimensional structures observed experimentally in shallow water flows are believed to appear as a result of hydrodynamic instability of the flow. From a practical point of view, complex eddy-type motion observed in experiments and in the field can lead to poor water quality since complex flows can trap sediments and pollutants. Linear stability analysis is often applied in such cases to determine conditions where a particular flow becomes unstable. Method of normal modes is used in[1,2,3] in order to obtain critical values of the bed-friction number below which shallow water flow becomes unstable with respect to small perturbations. Linear stability theory, however, cannot predict evolution of the most unstable mode above the threshold. Weakly nonlinear theories are used to study the development of instability when the value of the parameter characterizing the flow (for example, the Reynolds number for channel flows or bed-friction number for shallow flows) lies in the region of linear instability and is very close to the critical value on the stability boundary. Since the growth rate of an unstable mode in such cases is small, it is possible to take into account nonlinearities analytically by using weakly nonlinear expansions about the critical point. To the authors' knowledge, such an approach was considered for the first time in[4] for the case of a plane Poiseuille flow and was used later in[3,5] for shallow flows.

In the present paper we describe numerical methods for the solution of weakly nonlinear stability problems. Two examples are considered: (a) weakly nonlinear stability analysis of shallow water flows and (b) weakly nonlinear stability of two-component shallow flows for the case of large Stokes number.

2 GINZBURG-LANDAU EQUATION

Using methods of weakly nonlinear theory is shown in[3] that for the case of shallow water flows an amplitude evolution equation for the most unstable mode is the complex Ginzburg-Landau equation (CGLE). In the present paper we consider numerical method for calculation of the coefficients of the CGLE. However, for completeness basic steps of the derivation are summarized below. Consider shallow water equations under the rigid-lid assumtion of the form [3]

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} + \frac{c_f}{2h}u\sqrt{u^2+v^2} = 0, \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} + \frac{c_f}{2h}v\sqrt{u^2+v^2} = 0, \tag{3}$$

where u and v are the depth-averaged velocity components in the x and y directions, respectively, p is the pressure, c_f is the friction coefficient and h is water depth. Eliminating the pressure and introducing the stream function by the relations $u = \psi_y$, $v = -\psi_x$, we rewrite equations (1)–(3) in the form

$$\begin{aligned} (\Delta\psi)_t \;\; + \;\; & \psi_y(\Delta\psi)_x - \psi_x(\Delta\psi)_y + \frac{c_f}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2} \\ & + \frac{c_f}{2h\sqrt{\psi_x^2+\psi_y^2}}(\psi_y^2\psi_{yy} + 2\psi_x\psi_y\psi_{xy} + \psi_x^2\psi_{xx}) = 0. \end{aligned} \tag{4}$$

Consider a perturbed solution of (4) in the form

$$\psi(x,y,t) = \psi_0(y) + \varepsilon\psi_1(x,y,t) + \varepsilon^2\psi_2(x,y,t) + \varepsilon^3\psi_3(x,y,t) + \ldots. \tag{5}$$

Substituting (5) into (4) and neglecting the terms of order ε^2 and higher (the meaning of the parameter ε will be clarified later), we obtain the linearized equation of the form

$$L\psi_1(x,y,t) = 0, \tag{6}$$

where

$$L\psi_1 = \psi_{1xxt} + \psi_{1yyt} + \psi_{0y}(\psi_{1xxx} + \psi_{1yyx}) - \psi_{0yyy}\psi_{1x} + \frac{c_f}{2h}[(\psi_{1xx} + 2\psi_{1yy})\psi_{0y} + 2\psi_{1y}\psi_{0yy}] \quad (7)$$

Using the method of normal modes we assume that

$$\psi_1(x, y, t) = \varphi_1(y)\exp[ik(x - ct)], \quad (8)$$

where k is the wavenumber, c is the (complex) phase speed and $\varphi_1(y)$ is the amplitude of the normal perturbation. Substituting (8) into (6) we obtain

$$L_1\varphi_1 = 0, \quad (9)$$

where

$$L_1\varphi_1 = \varphi_{1yy}[ik(u_0 - c) + Su_0] + Su_{0y}\varphi_{1y} - \varphi_1[ik^3(c - u_0) - iku_{0yy} - k^2u_0S/2]. \quad (10)$$

Here $S = c_f b/h$ is the stability parameter and b is characteristic length scale (in case of wake flows b is the half-width of the wake [3]). The boundary conditions are

$$\varphi_1(\pm\infty) = 0. \quad (11)$$

For the given base flow velocity profile $u_0(y)$ eigenvalue problem (9), (11) can be solved numerically and the critical values of the parameters k, c and S, namely, k_c, c_c and S_c can be determined (see, for example,[2,3]).

In order to analyze the development of the most unstable mode in a weakly nonlinear regime we assume that the parameter S is slightly smaller than the critical value, namely, $S = S_c(1 - \varepsilon^2)$. Following[4] we introduce the "slow" time and longitudinal coordinate by the relations $\tau = \varepsilon^2 t$, $\xi = \varepsilon(x - c_g t)$, where c_g is the group velocity. Subsituting (5) into (4) and collecting the terms of orders ε^2 and ε^3 we obtain the following equations

$$L\psi_2 = g_1, \quad (12)$$

$$L\psi_3 = g_2. \quad (13)$$

In addition, the function $\psi_1(x, y, t)$ is sought in the form

$$\psi_1 = A(\xi, \tau)\varphi_1(y)\exp[ik_c(x - c_c t)] + c.c., \quad (14)$$

where the notation "c.c" denotes complex conjugate and A is the slowly varying amplitude of the most unstable mode. Functions g_1 and g_2 are bulky and for this reason are not shown here (the details can be found in [3]). Note that the function g_1 depends on ψ_1. The form of the function g_1 suggests that the solution to (12) should be sought in the form

$$\psi_2 = AA^*\varphi_2^{(0)}(y) + A_\xi\varphi_2^{(1)}(y)\exp[ik_c(x - c_c t)] + A^2\varphi_2^{(2)}(y)\exp[2ik_c(x - c_c t)] + c.c., \quad (15)$$

where A^* denotes the complex conjugate of A. Substituting (15) into (12) and replacing ψ_1 with (14) we obtain the following three boundary value problems for the unknown functions $\varphi_2^{(0)}(y)$, $\varphi_2^{(1)}(y)$ and $\varphi_2^{(2)}(y)$:

$$\begin{aligned} 2S_c(u_0\varphi_{2yy}^{(0)} + u_{0y}\varphi_{2y}^{(0)}) &= ik_c(\varphi_{1y}\varphi_{1yy}^* - \varphi_{1y}^*\varphi_{1yy} + \varphi_1\varphi_{1yyy}^* - \varphi_1^*\varphi_{1yyy} \\ &\quad - \frac{1}{2}S_c]k_c^2(\varphi_1\varphi_{1y}^* + \varphi_1^*\varphi_{1y}) + 2(\varphi_{1y}^*\varphi_{1yy} + \varphi_{1yy}^*\varphi_{1y}), \end{aligned} \tag{16}$$

$$\varphi_2^{(0)}(\pm\infty) = 0, \tag{17}$$

$$\begin{aligned} \varphi_{2yy}^{(1)}[ik_c(u_0 - c_c) + u_0S_c] &+ \varphi_{2y}^{(1)}u_{0y}S_c + \varphi_2^{(1)}[ik_c^3(c_c - u_0) - ik_cu_{0yy} - k_c^2u_0S_c/2] \\ &= \varphi_1(3k_c^2u_0 - 2k_c^2c_c + u_{0yy} - k_c^2c_g - ik_cu_0S_c) \\ &\quad + (c_g - u_0)\varphi_{1yy}, \end{aligned} \tag{18}$$

$$\varphi_2^{(1)}(\pm\infty) = 0, \tag{19}$$

$$\begin{aligned} [u_0S_c + 2ik_c(u_0 - c_c)]\varphi_{2yy}^{(2)} &+ u_{0y}S_c\varphi_{2y}^{(2)} - [8ik_c^3(u_0 - c_c) + 2ik_cu_{0yy} + 2k_c^2u_0S_c]\varphi_2^{(2)} \\ &= ik_c(\varphi_1\varphi_{1yyy} - \varphi_{1y}\varphi_{1yy}) - \frac{1}{2}S_c(2\varphi_{1y}\varphi_{1yy} - 3k_c^2\varphi_{1y}\varphi_{1yy}, \end{aligned} \tag{20}$$

$$\varphi_2^{(2)}(\pm\infty) = 0. \tag{21}$$

The operator L in (6), (12) and (13) is the same. Note that homogeneous equation (6) has a nontrivial solution. Then, in accordance with the Fredholm's alternative[6] the corresponding non-homogeneous equations (12) and (13) have solutions if and only if the right-hand sides of (12) and (13) are orthogonal to all eigenfunctions of the homogeneous adjoint problem. The adjoint operator L_1^a and adjoint eigenfunction φ_1^a of L_1^a are defined as follows:

$$\int_{-\infty}^{+\infty} \varphi_1^a L_1(\varphi_1)\, dy = \int_{-\infty}^{+\infty} \varphi_1 L_1^a(\varphi_1^a)\, dy. \tag{22}$$

The group velocity c_g can be found by applying the solvability condition to equation (12). Finally, using the solvability condition to equation (13) the amplitude evolution equation for the most unstable mode is obtained. It is shown in[3] that in this case the evolution equation is the complex Ginzburg-Landau equation (CGLE) of the form

$$\frac{\partial A}{\partial t} = \sigma A + \delta \frac{\partial^2 A}{\partial \xi^2} - \mu |A|^2 A, \tag{23}$$

where σ, δ and μ are complex coefficients. To summarize, in order to find the coefficients of the CGLE (23) one needs to perform the following calculations: (1) for a given base flow velocity profile $u_0(y)$ compute the critical values of the parameters k, S and c of linear stability problem (9), (11); (2) find the corresponding eigenfunction $\varphi_1(y)$; (3) compute the corresponding adjoint eigenfunction $\varphi_1^a(y)$; (4) solve three boundary value problems (16)–(21); (5) find the group velocity c_g from the solvability condition applied to equation (12); (6) evaluate the integrals from the solvability condition applied to equation (13) and compute the coefficients of the CGLE (23). Numerical procedure that is used for the calculation of the coefficients of the CGLE is desribed in detail in the next section.

3 NUMERICAL METHOD AND RESULTS

In order to solve linear stability problem (9), (11) we use a collocation method based on Chebyshev polynomials[7] that are defined on the interval [-1,1]. Hence, we map the interval $(-\infty, +\infty)$ by means of the transformation $r = 2/\pi \arctan y$. The function φ_1 in (9) is sought in the form

$$\varphi_1(r) = \sum_{k=0}^{N-1} a_k (1 - r^2) T_k(r), \tag{24}$$

where $T_k(r) = \cos k \arccos r$ are the Chebyshev polynomials of the first kind of order k, a_k are unknown koefficients and N is the number of collocation points on the interval $-1 \leq r \leq 1$. The factor $1 - r^2$ in (24) guarantees that zero boundary conditions at the endpoints of the interval [-1,1] are automatically satisfied. The collocation points inside the interval (-1,1)are given by

$$r_j = \cos \frac{\pi j}{N+1}, \quad j = 1, 2, \ldots, N-1. \tag{25}$$

Using the chain rule we compute the derivatives

$$\frac{d\varphi_1}{dy} = \frac{2}{\pi} \cos \frac{\pi r}{2} \frac{d\varphi_1}{dr}, \tag{26}$$

$$\frac{d^2\varphi_1}{dy^2} = \frac{4}{\pi^2} \cos^4 \frac{\pi r}{2} \frac{d^2\varphi_1}{dr^2} - \frac{4}{\pi} \sin \frac{\pi r}{2} \cos^3 \frac{\pi r}{2} \frac{d\varphi_1}{dr}, \tag{27}$$

$$\frac{d^3\varphi_1}{dy^3} = \frac{8}{\pi^3} \cos^6 \frac{\pi r}{2} \frac{d^3\varphi_1}{dr^3} - \frac{24}{\pi^2} \sin \frac{\pi r}{2} \cos^5 \frac{\pi r}{2} \frac{d^2\varphi_1}{dr^2} \tag{28}$$

$$+ \frac{1}{\pi} \left(-4 + 12 \tan^2 \frac{\pi r}{2} \right) \cos^6 \frac{\pi r}{2} \frac{d\varphi_1}{dr}, \tag{29}$$

where

$$\frac{d\varphi_1}{dr} = \sum_{k=0}^{N-1} a_k [-2r T_k(r) + (1 - r^2) T_k'(r)], \tag{30}$$

$$\frac{d^2\varphi_1}{dr^2} = \sum_{k=0}^{N-1} a_k [-2 T_k(r) - 4r T_k'(r) + (1 - r^2) T_k''(r)], \tag{31}$$

$$\frac{d^3\varphi_1}{dr^3} = \sum_{k=0}^{N-1} a_k [-6 T_k'(r) - 4r T_k''(r) + (1 - r^2) T_k'''(r)]. \tag{32}$$

Substituting (26), (27), (30), (31) into (9) and evaluating the function φ_1 and its derivatives at the collocation points (25) we obtain a generalized eigenvalue problem of the form

$$(A - cB)a = 0, \tag{33}$$

where A and B are complex-values matrices and $a = (a_0, a_1, \ldots, a_{N-1})$ is the vector containing unknown coefficients a_k. Numerical solution of (33) can be found, for example, by means of IMSL routine DGVCCG[8]. The same routine can be used to compute the corresponding eigenvector of (33) and the eigenfunction of problem (9), (11). Similarly one can solve the adjoint problem and find the corresponding eigenfunction φ_1^a.

Boundary value problems (16), (17) and (20), (21) are solved using (24). The corresponding systems of equations obtained after discretization are well-posed and can be solved by any linear equation solver. Problem (18), (19) is resonantly forced since the corresponding homogeneous problem has a nontrivial solution. As a result we use the singular value decomposition method[9] to solve problem (18), (19) after discretization. Finally, using the solvability condition for equation (14) the coefficients of the CGLE (23) are calculated as integrals containing functions φ_1, φ^a, $\varphi_2^{(0)}$, $\varphi_2^{(1)}$ and $\varphi_2^{(2)}$.

Two examples of weakly nonlinear analysis are presented below in order to illustrate the procedure. First, we consider shallow wake flow of one-component fluid governed by equation (4). The base flow velocity profile is chosen in the form

$$u_0(y) = 1 + \frac{2R}{1-R}\frac{1}{\cosh^2(\alpha y)}, \tag{34}$$

where $R = (U_c - U_a)/(U_c + U_a)$ is the velocity ratio, U_c is the velocity at the centerline, U_a is the ambient velocity, $\alpha = \sinh^{-1}(1)$.

Second, a two-component shallow flow is considered under the assumption that the Stokes number is considerably larger than unity. This assumption implies that heavy particles are uniformly distributed in the carrier fluid and are completely unresponsive to the changes in fluid motion. In other words, the particles are "frozen" in their initial state[10]. It is shown in[11] that for this case the amplitude evolution equation for the most unstable mode is also CGLE of the form (23). Using the formulas derived in[3] and[11] we present here the results of numerical computations of the coefficients of the CGLE (23). All computations are done for the velocity profile of the form (34)for the case $R = -0.5$ for three values of the particle loading parameter B[10] (see Table 1). As can be seen from Table 1, the real part of μ (referred to as the Landau constant in the literature) is positive for all cases considered so that the final amplitude saturation is possible.

4 CONCLUSIONS

Numerical aspects of weakly nonlinear theory are described in the present paper. Chebyshev collocation method is used to solve linear stability problem and corresponding adjoint problem. In addition, three boundary value problems are solved by the same

B	σ	δ	μ
0.0	0.0899+0.0004i	0.1150-0.1834i	4.5212+11.6033i
0.02	0.0716+0.0001i	0.1116-0.2131i	4.8302+11.7427i
0.04	0.0529-0.0000i	0.1062-0.2438i	5.3386+11.6620i

Table 1: The coefficients of the Ginzburg-Landau equation.

method (one of the problems is resonantly forced, therefore, singular value decomposition method is used to obtain the solution). Finally, the coefficients of the Ginzburg-Landau equation are calculated as integrals containing the five functions obtained before (the eigenfunctions of the linear stability problem and adjoint problem, and the three functions representing the solutions of the three boundary value problems). The complex Ginzburg-Landau equation can be used to analyze the development of the most unstable mode in a weakly nonlinear regime. Numerical calculations using the CGLE are in qualitative agreement with the results of numerical simulations of shallow wake flow behind an obstacle in[12] in terms of overall behavior of perturbations and saturation amplitude of the most unstable mode. It is also shown in[11] that purely periodic solutions of the CGLE are unstable and, therefore, are not observable. This fact is also consistent with experimental observations of shallow wake flows. However, a full spatio-temporal analysis of shallow wake flows in a nonlinear regime is required for verification of the Ginzburg-Landau model.

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