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ERROR ESTIMATE AND CONTROL IN THE MSFV METHOD FOR MULTIPHASE FLOW IN POROUS MEDIA

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Summary. In this paper the iterative MSFV method is extended to include the sequential implicit simulation of time dependent problems involving the solution of a system of pressure-saturation equations. To control numerical errors in simulation results, an error estimate, based on the residual of the MSFV approximate pressure field, is introduced. In the initial time steps in simulation iterations are employed until a specified accuracy in pressure is achieved. This initial solution is then used to improve the localization assumption at later time steps. Additional iterations in pressure solution are employed only when the pressure residual becomes larger than a specified threshold value. Efficiency of the strategy and the error control criteria are numerically investigated. This paper also shows that it is possible to derive an a-priori estimate and control based on the allowed pressure-equation residual to guarantee the desired accuracy in saturation calculation.

1 INTRODUCTION

Multiphase flow in a large geological formation, which honors integrated data provided by geoscientists, is computationally too expensive to be solved on the fine grid that defines detailed medium properties. In the past decade, several multiscale methods have been devised to reduce the computational cost[1-4]. Among the proposed methods, the Mixed Multiscale Finite Element (MMSFE)[2,4] and the Multiscale Finite Volume (MSFV)[3] methods provide locally conservative velocity fields, which makes them suitable for solving transport problems accurately.

For a wide range of heterogeneous test cases the solutions provided by the MSFV method are in very close agreement with those obtained from solving the problem on the original fine grid with a classical finite volume scheme. The accuracy of MSFV, however, deteriorates in

solving flow in highly channelized permeability fields (e.g. the SPE10 bottom-layer[5]) and highly anisotropic problems (e.g. heterogeneous test cases with stretched grids[6]). In order to improve the accuracy of the multiscale solution, several iterative MSFV (i-MSFV) methods have been proposed, which converge to the fine-scale reference result[7,8,10]. An appealing property of these i-MSFV methods is that a locally conservative fine-scale velocity field can be reconstructed at any iteration. This allows us to choose an appropriate number of iterations compatible with the requirements of accuracy and computational efficiency without compromising the flux conservations.

In this work the i-MSFV method is applied to solve the pressure equation in sequentially implicit simulation of multiphase flow. The transport equation in fine-scale is solved with the approximate velocity field, calculated by i-MSFV iterations. Since these approximate velocities are the only source of errors in solving the saturation (transport) equations, we can derive a formula to estimate this type of error. From error analysis, we show that controlling the residual reduces the error in most regions of the domain, except those close to the front with a sharp saturation gradient. Therefore, we propose to control the residual of the pressure equation in order to reduce the error of the coupled flow-transport system of equations. We numerically investigate the efficiency of the i-MSFV method for sequentially fully implicit simulations of multiphase flow in porous media.

2 GOVERNING EQUATIONS AND SIMULATION STRATEGY

From the mass conservation equations of n incompressible phases and the constraint that the pore volume is constant, we obtain the elliptic pressure equation

$$-\nabla\cdot\left(\mathbf{K}\lambda\nabla p\right)=q, \tag{1}$$

where K is the absolute permeability tensor, λ is the total mobility, and q is the source term. This pressure equation has to be solved together with n-1 saturation transport equations

$$\frac{d}{dt}\left(\phi S_\alpha\right)+\nabla\cdot\left(f_\alpha\mathbf{u}_t\right)=q_\alpha \qquad \alpha\in\{1,\ldots,n-1\}, \tag{2}$$

and subject to proper initial and boundary conditions. In equation (2), ϕ is the porosity, S_α is the phase saturation, f_α is the phase fractional flow function, and $\mathbf{u}_t=-K\lambda\nabla p$ is the total velocity vector.

Equation (1) is discretized by a standard 5-point-stencil finite volume scheme on a Cartesian grid. The resulting linear system can be written in the form

$$Ap=r, \tag{3}$$

where A is the coefficient matrix, p the unknown vector, and r the right hand side vector. Since inverting the penta-diagonal matrix is computationally too expensive for a large system, equation (3) is approximated by the MSFV method, which solves this linear equation more efficiently. First, a coarse grid is imposed on the domain and a dual (coarse) grid is constructed by connecting the centers of coarse cells. The dual grid naturally defines a partition of the fine cells into internal, edge, and node cells; and the system in equation (3) is

reordered such that the equations of internal cells appear first, then edge cells, and finally node cells[9]. The re-ordered system

$$\tilde{A}\tilde{p} = \tilde{r} \tag{4}$$

is obviously equivalent to the original system[9].

2.1 The iterative MSFV method

Instead of solving equation (4) directly, the MSFV method solves an approximate re-ordered system

$$M\tilde{p}' = Q\tilde{r}\,, \tag{5}$$

where the matrix M is block upper-triangular and is, therefore, cheaper to invert than $\tilde{A}$. Equation (5) describes both the solution of the coarse-grid balance problem and the subsequent calculation of edge and node pressures. The coarse grid pressure equation is constructed by means of basis and correction functions, and yields the node-point pressures. Subsequently reduced problems along the edges are solved using the node-point pressures as Dirichlet boundary conditions. And with the edge-point pressures as Dirichlet boundary conditions, the internal-point pressures are computed. This procedure is illustrated in figure 1. The matrix Q is a lower triangular matrix that replaces the node source terms by the appropriate right hand sides for the coarse-grid balance problem (and leaves the internal and edge source terms unchanged). This formulation of the MSFV method is described in detail by Lunati and Lee[9] and it is equivalent to the standard formulation of the MSFV method with correction function[11].

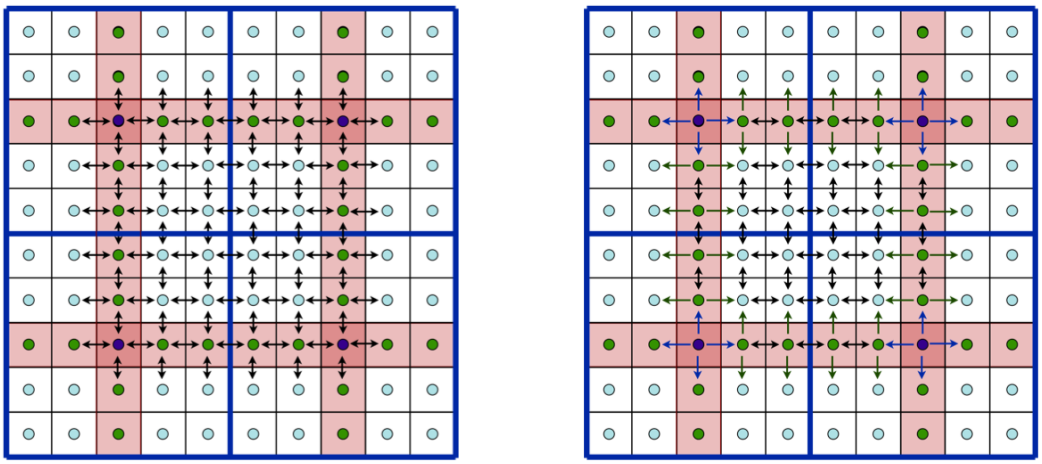

Figure 1: 2D domain with 10x10 fine grid (black lines) and 2x2 coarse grid (bold blue lines). In the partition introduced by the dual grid, fine cells are labeled as internal (light-blue), edge (green), and node (dark-blue) cells. Arrows are used to represent the stencil used for the fine-scale solution (left) and for the MSFV solution (right): denoting the arrow starting points as "predecessors" and the end points as "successors", an arrow indicates that the pressure value of the predecessor affects the mass balance of the successor (see Lunati and Lee[9]).

Since $\tilde{A}$ and M are different, the fine-scale solution and the MSFV solution are never strictly identical. This is due to the localization operated along the edges of the dual cells, which neglects the transversal fluxes[7,8,10]. An iterative algorithm converging to the fine-scale solution, however, can be constructed estimating the neglected fluxes from the residual $(\tilde{r} - \tilde{A}\tilde{p})$, where $\tilde{p}$ is the solution of equation (5). After some mathematical manipulations, one obtains an iterative scheme of the form[10]

$$\tilde{p}^{\nu+1} = \tilde{p}^{\nu} + w^{\nu} M^{-1} Q(\tilde{r} - \tilde{A}\tilde{p}^{\nu}), \tag{6}$$

where ν is the iteration superscript (and initial pressure value is $\tilde{p}^0 = M^{-1}Q\tilde{r}$), and w^{ν} is a relaxation parameter, which is computed by the Generalized Minimal Residual (GMRES) method[12] to accelerate convergence.

2.2 The i-MSFV method for time dependent problems

An important property of the i-MSFV method is that the velocity field $\tilde{u}_t = -K\lambda_t \nabla \tilde{p}^{\nu+1}$ (at any iteration level ν) is conservative at the coarse scale. To obtain a conservative velocity field at the fine scale, additional local problems are solved in the coarse cells subject to $\tilde{u}_t$ as Neumann boundary condition, resulting in the reconstructed velocity field $\tilde{u}_t'' = -K\lambda_t \nabla \tilde{p}''$. The approximate velocity field u_t, which is equal to $\tilde{u}_t''$ inside the coarse cells and to $\tilde{u}_t$ on the coarse-cell boundaries, is locally conservative, i.e. $\nabla . u_t = 0$, at the fine-scale[13]. Therefore, the conservative velocity field is used to solve the nonlinear saturation equations (2) on the fine grid. Implicit time integration leads to the nonlinear saturation equation,

$$\frac{\phi}{\Delta t}\left(S_\alpha^{n+1} - S_\alpha^n\right) + \nabla \cdot \left(f_\alpha^{n+1}\mathbf{u}_t\right) = q_\alpha, \tag{7}$$

which is solved by Newton Raphson iterations.

After convergence of this iterative loop (inner loop), the mobility is updated with the new saturation, S_α^{n+1}, and then the pressure equation is solved again. This procedure is repeated until the coupled pressure-saturation (or flow-transport) system converges (outer loop). In this solution scheme, the pressure equation is solved several times. To improve the efficiency of numerical simulation, it is important to minimize the number of iterations in (6). This can be achieved by using the old pressure solution to estimate the transversal fluxes across the dual edges in the new-pressure solution[7]. This corresponds to use the solution of the equation

$$M\tilde{p}^0 = Q\tilde{r} + E(M\tilde{p}^n - Q\tilde{r}), \tag{9}$$

as an initial pressure value for the iterative scheme in equation (6). The second term on the right hand side computes the estimated gradient of transversal fluxes from the old pressure solution, and E is an appropriate matrix operator[10].

3 ERROR CONTROL OF THE SATURATION EQUATION

As mentioned before, the stopping criterion for i-MSFV iterations entails a quality measure of the pressure field. Since the saturation equations are solved at fine-scale, the error in the

phase saturation field is only incurred due to the approximate pressure field from the incompletely converged solution in the i-MSFV method. The fine-scale saturation transport equation for phase α given by equation (2) can be restated as

$$\frac{d}{dt}(\phi S_\alpha) + f_\alpha \nabla . u_t^* + u_t^* . \nabla f_\alpha + f_\alpha \nabla . \hat{u}_t + \hat{u}_t . \nabla f_\alpha = f_\alpha q, \tag{10}$$

where the superscripts * and ^ stand for the exact and error terms in the total velocity, i.e. $u_t = u_t^* + \hat{u}_t$. The reformulation of the saturation equation will consequently lead to the error terms introduced by the inexact velocity field, i.e.

$$e = f_\alpha \nabla \cdot \hat{u}_t + \hat{u}_t . \nabla f_\alpha = f_\alpha \nabla \cdot \hat{u}_t + \hat{u}_t . (df_\alpha / dS_\alpha) \nabla S_\alpha . \tag{11}$$

As previously mentioned, in the MSFV and i-MSFV methods, the approximate velocity field u_t is calculated based on the reconstructed ($\tilde{p}''$) and superimposed ($\tilde{p}^{\nu+1}$) pressure fields in a way that it is divergence free, i.e. $\nabla . u_t = 0$ [13]. The exact velocity field is also divergence free, i.e. $\nabla . u_t^* = 0$. As a result, the error in the saturation equation is equal to $\hat{u}_t . \nabla f_\alpha$ that is introduced due to the approximate conservative i-MSFV velocity u_t.

Direct calculation of the velocity error $\hat{u}_t$ is not possible, since it requires a priori knowledge of the reference solution. On the other hand, the quality of the conservative velocity u_t depends only on the quality of the local Neumann boundary conditions provided by the non-conservative i-MSFV velocity field, $\tilde{u}_t = -K\lambda_t \nabla \tilde{p}^{\nu+1}$. As mentioned before, the error of $\tilde{u}_t$ is only due to the neglected transversal fluxes, which are equivalent to unphysical source terms at the boundaries of dual cells. Therefore, the quality of $\tilde{u}_t$ can be estimated from the residual of the non-conservative pressure field $\tilde{p}^{\nu+1}$, i.e.

$$\Re = (\tilde{r} - \tilde{A}\tilde{p}^{\nu+1}). \tag{12}$$

Based on the phase saturation distribution in the domain, the velocity error estimated by (12) can have a minor or major effect on the phase saturation error. For example, in the solution of the 2-phase Buckley-Leverett equation, two different zones are identified: (1) near front areas where the saturation gradients are high, and (2) behind the front area where the saturation gradients are low. In the i-MSFV framework, very accurate total velocities are required only at the front region, which is usually a very small sub-domain; while, for the rest of the domain, the velocity field of moderate quality can result in accurate saturation solutions. Therefore, it is very important to obtain an efficient strategy in the i-MSFV framework to define an effective pressure-residual-based criterion to guarantee the required accuracy in phase-saturation. This will be further investigated in our future work.

Finally, it is worth mentioning that if a non-conservative velocity field is used to solve the saturation equation, the term $f_\alpha \nabla \cdot \hat{u}_t$ in equation (11) is not zero. Therefore, to obtain accurate saturation solutions, very accurate total velocity fields are required not only at a very small sub-domain close to the saturation front, but also in a relatively big sub-domain, where the value of fractional flow function is significant.

4 NUMERICAL RESULTS

A quarter of five-spots problem is solved in a 2D heterogeneous anisotropic medium (see Figure 2). The permeability in vertical direction is 20 times more than that of horizontal direction, i.e. $k_y=20k_x$; the computational grid contains 55x55 fine and 5x5 coarse cells. Gas is injected at constant rate of 10 in the lower left corner, while the upper-right corner is kept at constant pressure, $p=0$. In this example gravity is neglected. The viscosity ratio is 10 and the quadratic relative permeability curves are employed. Figure 3 shows the fine-scale reference saturation and pressure maps at 0.2 PVI, and figure 4 shows the pressure and gas saturation errors in the MSFV and i-MSFV solutions, which are defined as the absolute differences with respect to the fine-scale results. Pressure errors are normalized by the fine-scale pressure at the injection point. For this test case the original MSFV method fails to give a good result. The i-MSFV method, however, leads to very accurate results with only 0.7 iterations in average per each pressure solver call (i.e., the computational cost of i-MSFV is only 1.7 times that of the classical MSFV method). Figure 5 illustrates the history of the number of iterations employed, and figure 6 depicts the saturation errors for different residual threshold values.

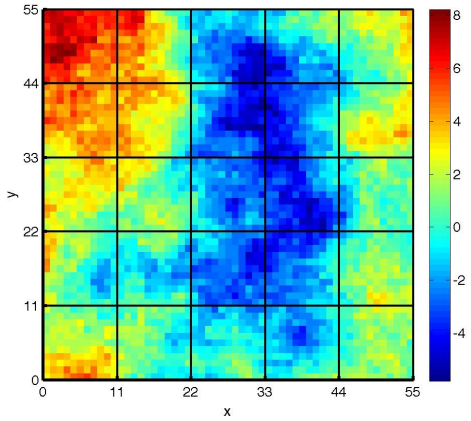

Figure 2: Natural logarithm distribution of the horizontal permeability k_x with the imposed 5x5 coarse grid.

5 CONCLUSIONS

The i-MSFV method is extended to include the sequential implicit simulations of multiphase flow in heterogeneous anisotropic porous media. In this work, the i-MSFV method is only used for the efficient solution of the elliptic pressure equation, whereas the saturation equation is solved on the fine scale. At the beginning of the simulation, iterations are employed to improve the MSFV solutions. To minimize the need for further iterations, this initial solution is used to improve the MSFV system at later steps. If the pressure residual becomes larger than a specific threshold value, additional iterations are employed to reduce the residual. The numerical tests show that only a few iterations are required during the simulation to significantly improve the MSFV results (and in particular the saturation solutions) with a relatively small additional computational cost. The overall efficiency of the

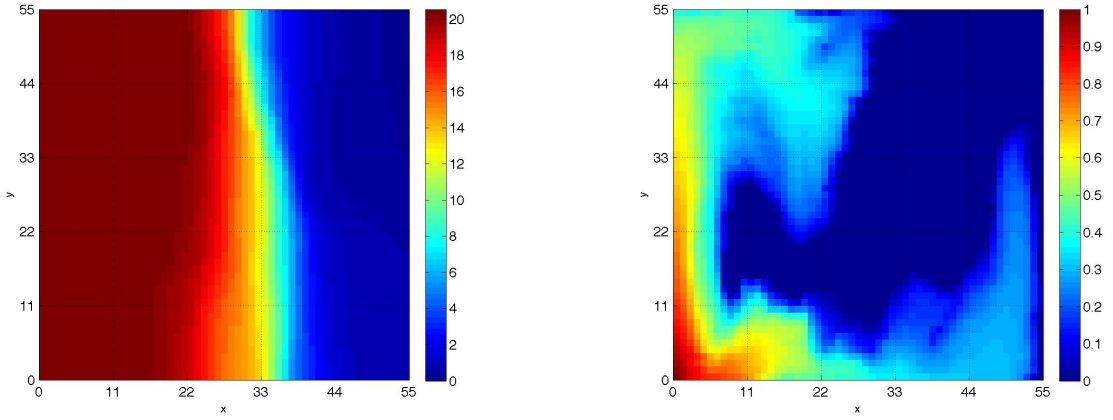

Figure 3: Fine scale reference pressure (left) and gas saturation (right) maps after 0.2 PVI.

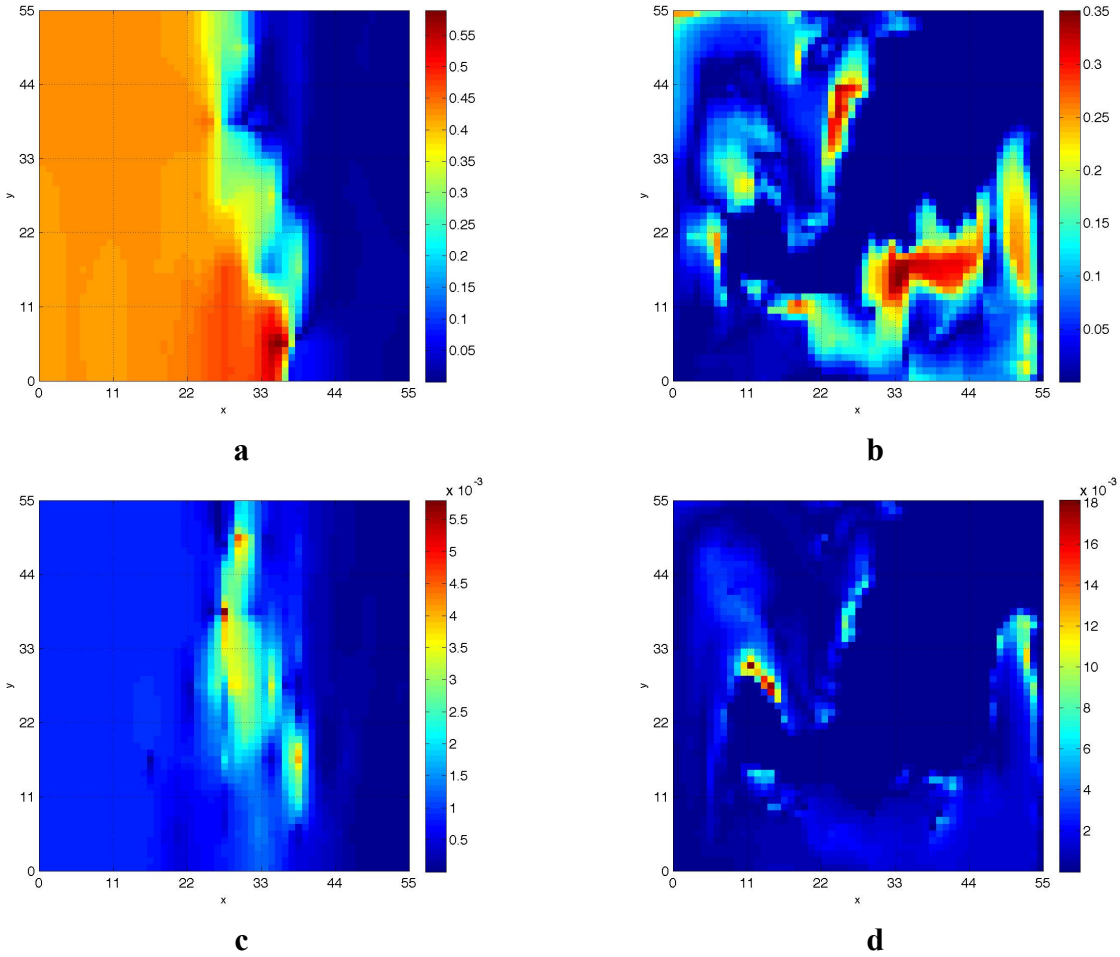

Figure 4: Normalized pressure (**a**) and gas saturation (**b**) errors in MSFV; and normalized pressure (**c**) and gas saturation (**d**) errors in i-MSFV. The simulation time is 0.2 PVI and the i-MSFV results are obtained by employing 0.7 iterations in average.

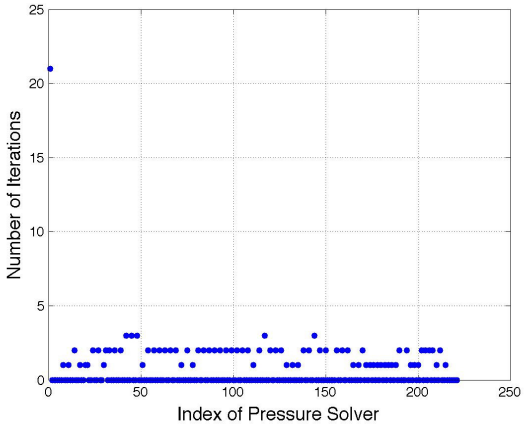

Figure 5: History of i-MSFV iterations employed during the simulation. In average, 0.7 iterations are employed.

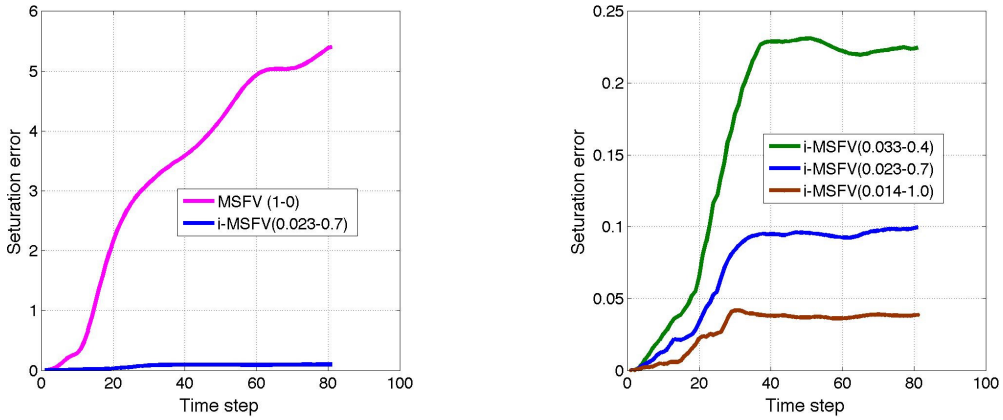

Figure 6: L2 norm of the saturation errors during the simulation time corresponding to the results of figure 4 (left) and the i-MSFV results with different residual threshold values (right). The first number in the parentheses denotes the average L2 norm of the residual (normalized by that of the MSFV residual) and the second one denotes the average number of iterations employed during the simulation.

simulations can be further improved by solving also the transport equations with the MSFV method[14], and by using the adaptive MSFV residual improvement[15]. In the latter case, one can define more strict threshold values near front regions and moderate ones for the rest of the domain.

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