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APPLICATIONS OF SMOOTHED PARTICLE HYDRODYNAMICS TO ENGINEERING FLUID DYNAMICS

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Summary. This paper describes how Smoothed Particle Hydrodynamics can be applied to problems involving fluids and solids where there may be more than one phase, and more than one material. The advantage of replacing continuum fluids by moving particles is that free surfaces and splash require no special treatment. In addition the advection of material is treated very accurately. Examples of its application include fluids in containers with complicated geometry, free surfaces, and moving bodies.

1 INTRODUCTION

Smoothed particle hydrodynamics (SPH) is a numerical method which replaces a fluid or an elastic body by a set of particles. The particles may be thought of either as real particles, or as moving interpolation points. The key feature of the method is that spatial derivatives of quantities can be estimated without using a grid by means of a technique known as kernel estimation. The derivatives can be found from particle information even though the positions of the particles may be disordered. As a result, free surface problems such as breaking waves present no problems. The method is easily extended to more than one fluid or to bodies floating in the fluid. Another attractive aspect of SPH is in the simulation of mixing. If a scalar property is assigned to some of the particles, for examples chemical type, they will carry that property with them. As a result advection is treated very accurately. Rigid bodies moving in the fluid can also be treated in a straight forward way. In the case of fracture SPH has been shown to be very effective because it can move seamlessly from the continuum to a set of fragments. Many of the details which cannot be covered in this paper are given in reviews [4,5].

2 KERNEL ESTIMATION

We begin with a set of particles each with some property A and our aim is to estimate $A(\mathbf{r})$ for any point $\mathbf{r}$ using the values of A and the coordinates of each particle. We start with the following exact result for any quantity $A(\mathbf{r})$ which may be a scalar, vector or tensor

$$A(\mathbf{r}) = \int A(\mathbf{r}'\delta(\mathbf{r}-\mathbf{r}')\mathbf{dr}', \tag{1}$$

where $\mathbf{dr}$ denotes an element of volume, and the Delta function $\delta(\mathbf{q})$ has the following properties

$$\int \delta(\mathbf{q})\mathbf{dq} = 1. \tag{2}$$

$\delta(\mathbf{q})$ is zero except where $|\mathbf{q}| = 0$. While this result is exact we cannot use it directly because functions of this form cannot be integrated numerically. Instead we replace the Delta function by a smooth function $W(\mathbf{q}, h)$ which becomes a Delta function when $h \to 0$ and satisfies the condition

$$\int W(\mathbf{q}, h)\mathbf{dq} = 1, \tag{3}$$

There are infinitely many such functions [4,5,7] but it helps to keep in mind a Gaussian in one dimension which has the form

$$\frac{1}{h\sqrt{\pi}}e^{-q^2/h^2}. \tag{4}$$

The function W is called the kernel. We can now estimate $A(\mathbf{r})$ by

$$A(\mathbf{r}) = \int A(\mathbf{r}')W(\mathbf{r}-\mathbf{r}')\mathbf{dr}'. \tag{5}$$

In order to evaluate this integral we replace the volume integration by an integration over mass elements dm using $dm = \rho\mathbf{dr}$, where ρ is the density. We then get the following summation interpolant for the function A

$$A(\mathbf{r}) = \sum_a \frac{m_a}{\rho_a} A(\mathbf{r_a})W(\mathbf{r}-\mathbf{r}_a, h), \tag{6}$$

where the summation is over all the fluid particles, and a denotes a particle label. The interpolant estimates $A(\mathbf{r})$ by a function which is analytic if the kernel W is an analytic function. As a consequence derivatives of the function A can be calculated exactly. A discussion of how this is done for the equations of fluid dynamics is given in the review articles cited earlier.

3 EQUATIONS OF MOTION

The acceleration equation can be obtained directly from the equations of fluid dynamics or by using a Lagrangian. Whichever of these approaches is used the Euler equation for a particle a in a homogeneous fluid in the absence of boundaries takes the form

$$\frac{d\mathbf{v}_a}{dt} = -\sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} \right) \nabla_a W_{ab}, \tag{7}$$

and the continuity equation becomes

$$\frac{d\rho_a}{dt} = \sum_b m_b (\mathbf{v}_b - \mathbf{v}_b) \cdot \nabla_b W_{ab}. \tag{8}$$

where W_{ab} denotes $W(\mathbf{r}_a - \mathbf{r}_b, h)$. A slightly different form of these equations is used when there is more than one fluids with very different densities, as in the case of air and water, or fresh and salty water. These equations can be integrated with methods which have been worked out for molecular dynamics. The SPH equations conserve linear and angular momentum when they should and they approximately conserve a discrete version of the circulation theorem. Provided a time stepping scheme such as the Verlet symplectic method the energy is conserved to second order in the time step.

We have left aside the question of how to calculate P. The original application of SPH to nearly incompressible fluids such as water used a slightly compressible equation of state with a speed of sound below the real speed of sound but large enough to ensure the density fluctuations were small, typically $\sim 1\%$. This can be achieved by taking the speed of sound to be 10 times the largest fluid speed. A number of studies have discussed exactly incompressible SPH [(2,8)]

3.1 Viscous forces

Viscosity can be included by replacing (7) by

$$\frac{d\mathbf{v}_a}{dt} = -\sum_b m_b \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} + \Pi_{ab} \right) \nabla_a W_{ab}, \tag{9}$$

where Π_{ab} can take a number of forms but the following is typical

$$\Pi_{ab} = -\frac{8\nu \mathbf{v}_{ab} \cdot \mathbf{r}_{ab}}{\bar{\rho}_{ab} h |\mathbf{r}_{ab}|}, \tag{10}$$

where ν is the kinematic viscosity coefficient, and $\bar{\rho}_{ab} = \frac{1}{2}(\rho_a + \rho_b)$. If the system has several fluids with different viscosities then this viscous term is replaced by

$$\Pi_{ab} = -\frac{16\nu_a \nu_b \mathbf{v}_{ab} \cdot \mathbf{r}_{ab}}{(\nu_a \rho_a + \nu_b \rho_b) h |\mathbf{r}_{ab}|}, \tag{11}$$

This form of the viscosity is Galilean invariant (as are the original equations), it guarantees that the viscous dissipation is positive definite, and it vanishes for rigid rotation. The continuum equivalent [(3)] is a shear and a bulk viscosity.

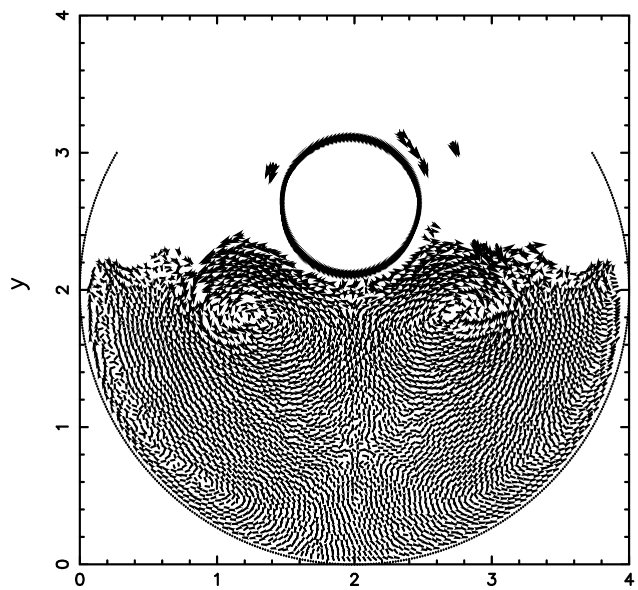

Figure 1: The particle positions and velocities of a cylinder rising inside a basin. Note the fluid SPH particles running off the rising cylinder and the vortices which form behind it.

3.2 Boundaries

A variety of techniques have been used for boundaries. These include ghost or image particles or boundary force particles. In the case of the latter the boundaries exert forces on the fluid in a manner similar to the technique known as the Immersed Boundary Method. For the examples shown here the acceleration equation becomes

$$\frac{d\mathbf{v}_a}{dt} = -\sum_{\eta} m_\eta \left(\frac{P_a}{\rho_a^2} + \frac{P_b}{\rho_b^2} + \Pi_{a\eta} \right) \nabla_a W_{a\eta} + \sum_j m_j \mathbf{r}_{aj} f(|\mathbf{r}_{aj}|), \tag{12}$$

where now the summation is over both the fluid SPH particles and the boundary force particles. The function f is a smooth function which decreases rapidly with distance from the boundary [6]

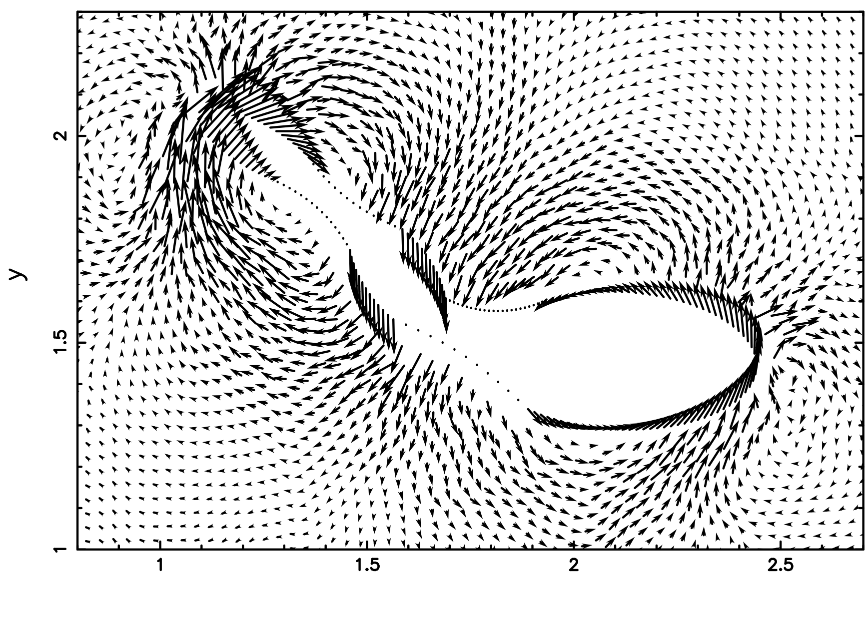

Figure 2: The particle positions and velocities of 3 linked bodies moving through a fluid with specified changes of the angles between the bodies. The velocity arrows of the particles start on the particles.

4 APPLICATIONS

In this section we show some characteristic SPH simulations which indicate the flexibility of the method.

4.1 Buoyant cylinder in a basin

We consider the rise of a buoyant cylinder in a basin [6]. The boundaries of the basin and the cylinder are specified by boundary force particles. The motion of the cylinder is determined by the forces exerted on it by means of the forces on its boundary force particles. Figure 1 shows the cylinder after it has floated up and through the free surface. The fluid spilling off the cylinder can be seen as can the two vortices on each side of the cylinder.

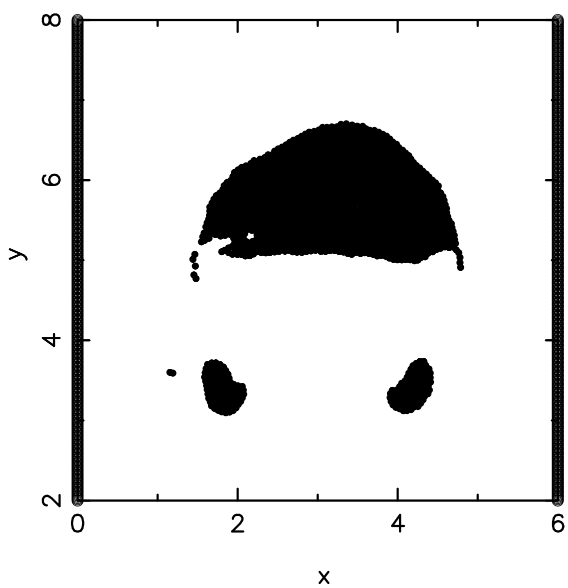

Figure 3: The positions of the SPH particles forming a bubble of radius 1m rising in a tank . The bubble is initially circular then deforms to produce the shape shown with small parts of the bubble left behind as trailing bubbles.

4.2 Swimming linked bodies

Linked bodies with specified time variation of the angles between them move through a fluid like a fish [(3)]. The present example shown in figure 2 involves three bodies of different size in the form of ellipses. The bodies are connected by an elastic skin formed of SPH skin particles which interact with the fluid and with the body. Because of the connection between the bodies the equations of motion involve constraints. In the present case these constraints were taken care of by using Lagrange multipliers. This problem is not only of interest to Zoologists working on marine animals but also to marine engineers designing underwater marine vehicles.

4.3 Large air bubbles in water

There have been a number of successful applications of SPH to air and water starting with [1]. A simpler version was introduced recently for large bubbles where surface tension can be neglected. An example is shown in figure 3 where a bubble intially in the form of a circle of radius 1m rises in a tank of width 6m and height 10m. The bubble eventually forms a shape like $\bigcap$ then the end sections break off and the remainder forms a semi circle.The results are in good agreement with level set calculations.

5 CONCLUSIONS

SPH has become a standard method for the solution of complex problems in fluid dynamics. It is readily extended to physical problems that involve multiple fluids and bodies and the change of thermodynamics state of the material. Examples of this latter problem are freezing salt solutions, the solidification of lava and the effects of surfactants on surface tension. References to these simulations can be found in the review articles, and further information can be found on the SPHERIC website. SPH has also become one of the most powerful techniques for the simulation of special effects in movies. The reader is invited to inspect the website of the Spanish company NextLimit (Madrid) that has used SPH for numerous movies.

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