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CENTRAL WENO FOR SHALLOW WATER EQUATIONS IN CONTRAVARIANT FORMULATION

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Summary. A new Central Weighted Essentially Non-Oscillatory scheme for the solution of the shallow water equations expressed directly in contravariant formulation is presented. The proposed central WENO scheme is the extension of the methodology presented by many authors[1,2,3,4] into the context defined by contravariant form of the equations. One of the most important elements of the C-WENO scheme based on this approach involves the advancing from time level t^n to time level t^{n+1} of the cell averaged values of flow variables. The extension of the above mentioned methodology into the contravariant environment implies that the contravariant shallow water equations must be expressed in integral form. An element of novelty presented in this paper regards the definition of a formal integral expression of the shallow water equations in contravariant formulation, in which Christoffel symbols are avoided. The WENO reconstructions are performed by a two dimensional interpolating procedure taking into account the curved coordinate lines; in the computational domain the spatial discretization step is constant: consequently the problems related to negative linear weights on unstructured meshes are overcome. The two dimensional reconstructions have a fifth-order spatial accuracy. A Natural Continuous Extension into a Runge-Kutta solver is involved in a fourth-order time discretization of motion equations. The proposed scheme ensures the satisfaction of the exact C-property.

1 INTRODUCTION

Many authors solve shallow water equations by using high-resolution methods for hyperbolic systems of conservation laws. In this context, Essentially Non-Oscillatory (ENO) and Weighted Essentially Non-Oscillatory (WENO)[5] schemes are the most efficient tools.

Flow simulations over computational domains characterized by a complex boundary can be performed by numerical integrations of motion equations on a generalized curvilinear boundary conforming grid. In this approach the domain is greatly simplified, since it is

transformed into a fixed rectangular region and the WENO reconstructions are performed directly in transformed space where the computational space step is constant. In order to make use of a general boundary-conforming curvilinear coordinate system, two different strategies can be followed: in the first strategy, motion equations are projected onto directions identified by the Cartesian reference system and are modified by transforming partial derivatives with respect to Cartesian coordinates to partial derivatives with respect to curvilinear coordinates; in the second strategy, motion equations are expressed directly in covariant or contravariant formulation.

The introduction of WENO schemes into the curvilinear coordinate context (defined by the first strategy) implies the carrying out of Cartesian based flow variable reconstructions on curved coordinate lines. When high order reconstructions are developed on very nonuniform grids a contradiction could appear: the strongly curved coordinate line on which the flow variable must be reconstructed could become almost orthogonal to the Cartesian direction, on which the flow variable is based.

The second strategy is represented by the possibility to express motion equations directly in the coordinate basis defined by the numerical grid. The equations become a system of balance laws with flux functions that depend explicitly on the spatial coordinates. In generalized curvilinear coordinates, contravariant components are vector components defined on a basis which is locally normal to the curvilinear coordinate lines.

In numerical solutions of motion equations in contravariant formulation, a contradiction appears. It is well known that a strong conservation form of convective terms can preserve freestream properties without any pollution due to grid skewness. In the contravariant formulation of motion equations, covariant derivatives give rise to Christoffel symbols. These terms are extra source terms. They come in with the variability of base vectors and do not permit the definition of convective terms in a strong conservation form. As a consequence, numerical discretization of the Christoffel symbols can reduce the numerical accuracy.

The original contribution of this work is the definition of a new Central Weighted Essentially Non-Oscillatory scheme for the solution of the shallow water equations expressed directly in contravariant formulation. The proposed central WENO scheme is the extension of the methodology presented by many authors[1,2,3,4] into the context defined by contravariant form of the equations. One of the most important elements of the C-WENO scheme based on this approach involves the advancing from time level t^n to time level t^{n+1} of the cell averaged values of flow variables. In order to perform the advancing in time of cell averaged values of the flow variables, Caleffi et al.[4] use the integral formulation of one-dimensional shallow water equations. Jiang et al.[2] perform the computation of cell averaged values of flow variables by the numerical discretization of two-dimensional cell integrated motion equations on a rectangular grid. Consequently the extension of the above mentioned methodology into the contravariant context implies that the contravariant shallow water equations must be expressed in integral form.

An element of novelty presented in this paper regards the definition of a formal in-

tegral expression of the shallow water equations in contravariant formulation in which Christoffel symbols are avoided. The WENO reconstructions are performed by a two dimensional interpolating procedure taking into account the curved coordinate lines; in the computational domain the spatial discretization step is constant: consequently the problems related to negative linear weights on unstructured meshes are overcome. The two dimensional reconstructions have a fifth-order spatial accuracy. A Natural Continuous Extension into a Runge-Kutta solver is involved in a fourth-order time discretization of motion equations. The proposed scheme ensures the satisfaction of the exact C-property.

2 SHALLOW WATER EQUATIONS IN CONTRAVARIANT FORMULATION

We define the water depth as h and the depth averaged velocity vector as $\vec{u}$, whose components are defined in the Cartesian system of reference. Let be $\vec{v} = \vec{u}h$. The shallow water equations are written directly in the contravariant formulation in a two-dimensional curvilinear coordinate system.

In order to introduce the notation to be used, we consider a transformation $x^l = x^l(\xi^1, \xi^2)$ from the Cartesian coordinates $\vec{x}$ to the curvilinear coordinates $\vec{\xi}$ (note that superscripts indicate components and not powers in the present notation). Let $\vec{g}_{(l)} = \partial\vec{x}/\partial\xi^l$ be the covariant base vectors and $\vec{g}^{(l)} = grad(\xi^l)$ the contravariant base vectors. The metric tensor and its inverse are defined by $g_{lm} = \vec{g}_{(l)} \cdot \vec{g}_{(m)}$ and $g^{lm} = \vec{g}^{(l)} \cdot \vec{g}^{(m)}$ $(l, m = 1, 2)$. The Jacobian of the transformation is given by $\sqrt{g} = \sqrt{det(g_{lm})}$. Let r^l be the contravariant components in the curvilinear coordinate system of the vector $\vec{v}$. The transformation relationships between vector $\vec{v}$ in the Cartesian coordinate system and its contravariant and covariant components, r^l and r_l, in the curvilinear coordinate system are[6]:

$$r^l = \vec{g}^{(l)} \cdot \vec{v} \quad , \quad \vec{v} = r^l \vec{g}_{(l)} \quad ; \quad r_l = \vec{g}_{(l)} \cdot \vec{v} \quad , \quad \vec{v} = r_l \vec{g}^{(l)} \tag{1}$$

The same relationship also applies to other vectors. In the following equations, a comma with an index in a subscript denotes covariant differentiation. The covariant derivative is defined as $u^l_{,m} = \partial u^l/\partial\xi^m + \Gamma^l_{mk} u^k$ where Γ^l_{mk} is the Christoffel symbol that is given by $\Gamma^l_{mk} = \vec{g}^{(l)} \cdot \partial\vec{g}_{(k)}/\partial\xi^m$.

The shallow water equations in contravariant formulation are

$$\frac{\partial h}{\partial t} + r^l_{,l} = 0 \tag{2}$$

$$\frac{\partial r^l}{\partial t} + \left(\frac{r^l r^m}{h}\right)_{,m} = -Ghg^{lm}(h + H)_{,m} - R^l \tag{3}$$

where $R^l = G(r^l|\vec{r}|)/(\chi^2 h^2)$ is the bed resistance term, $\chi = (1/M)h^{1/6}$ is the friction coefficient, M is the Manning friction factor, G is the constant of gravity and H is the

bed height and $\left(\frac{r^l r^m}{h}\right)_{,m}$ is the covariant derivative of $\frac{r^l r^m}{h}$, and is given by:

$$\left(\frac{r^l r^m}{h}\right)_{,m} = \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\frac{r^l r^m}{h}\right)}{\partial \xi^m} + \left(\frac{r^m r^k}{h}\right)\Gamma^l_{mk} \tag{4}$$

It has to be noticed that the simple integration of Equation (3) over an arbitrary surface element does not avoid Christoffel symbols. Our main goal is to formalize an integral expression of the contravariant shallow water equations in which Christoffel symbols are absent.

Let us integrate Equation (2) over an arbitrary surface element ΔA whose contour line is L:

$$\int_{\Delta A} \frac{\partial h}{\partial t} dA + \int_L r^m n_m dL = 0 \tag{5}$$

The second integral of Equation (5) has been transformed by Green's theorem and n_m is the covariant outward normal. In order to express the momentum conservation law in integral form, we take a constant parallel field of vectors λ_k and equate the rate of change of momentum of a material volume to the total net force in this direction: this is necessary since the direction in space of a given coordinate line is changing, in contrast to the Cartesian case.

We choose, as parallel vector field, the one which is normal to the coordinate line on which the ξ^l coordinate is constant at point $P_0 \in \Delta A$. The coordinate values of P_0 are ξ^1_0 and ξ^2_0. The contravariant base vector, which is normal to the coordinate line on which ξ^l is constant, at point P_0 is given by $\vec{g}^{(l)}(\xi^1_0, \xi^2_0)$. From Equation (1) we can deduce the transformation relationship between the vector $\vec{g}^{(l)}(\xi^1_0, \xi^2_0)$ and its covariant components $\lambda_k(\xi^1_0, \xi^2_0)$ in the curvilinear coordinate system:

$$\lambda_k(\xi^1_0, \xi^2_0) = \vec{g}^{(l)}(\xi^1_0, \xi^2_0) \cdot \vec{g}_{(k)}(\xi^1_0, \xi^2_0) = \delta^l_k \tag{6}$$

where δ^l_k is the Kronecker symbol.

Let $\lambda_k(\xi^1_0, \xi^2_0)$ be the vector which defines the parallel vector field. Let $\vec{g}_{(k)}(\xi^1, \xi^2)$ be the covariant base vector which is tangent to the ξ^k coordinate line at the generic point P whose coordinates are ξ^1 and ξ^2. The covariant components of the vector, which are expressed as a function of the above mentioned covariant base vector $\vec{g}_{(k)}$, are:

$$\lambda_k(\xi^1, \xi^2) = \vec{g}^{(l)}(\xi^1_0, \xi^2_0) \cdot \vec{g}_{(k)}(\xi^1, \xi^2) \tag{7}$$

For the sake of brevity, we indicate $\vec{g}^{(l)} = \vec{g}^{(l)}(\xi^1_0, \xi^2_0)$ and $\vec{g}_{(k)} = \vec{g}_{(k)}(\xi^1, \xi^2)$. We integrate over an arbitrary surface element ΔA and resolve in the direction λ_k the rate of change of the depth-integrated momentum (per unit mass) and the depth-integrated force (per unit mass). Consequently we get:

$$\int_{\Delta A} \frac{\partial r^k}{\partial t} \lambda_k dA + \int_{\Delta A} \left(\frac{r^k r^m}{h}\right)_{,m} \lambda_k dA = \int_{\Delta A} [Ghg^{mk}(h+H)_{,m} + R^k]\lambda_k dA \tag{8}$$

Since the field of vectors is parallel, $\lambda_{k,m} = 0$. By transforming the second integral on the left hand side of Equation (8) by Green's theorem and by using Equation (7) we get

$$\int_{\Delta A} \vec{\bar{g}}^{(l)} \cdot \vec{g}_{(k)} \frac{\partial r^k}{\partial t} dA + \int_{\Delta L} \vec{\bar{g}}^{(l)} \cdot \vec{g}_{(k)} \frac{r^k r^m}{h} n_m dL = \int_{\Delta A} \vec{\bar{g}}^{(l)} \cdot \vec{g}_{(k)} [Ghg^{mk}(h+H)_{,m} + R^k] dA \quad (9)$$

Equations (5) and (9) represent the integral expressions of the shallow water equations in contravariant formulation.

3 THE CENTRAL WENO SCHEME

The numerical integration of Equations (5) and (9) is performed by a central WENO scheme. The discretization of the computational domain is based on a grid defined by the coordinate lines ξ^1 and ξ^2 and by the points of coordinates $i\Delta\xi^1$ e $j\Delta\xi^2$, and on a second staggered grid, defined by the points of coordinates $\left(i+\frac{1}{2}\right)\Delta\xi^1$ and $\left(j+\frac{1}{2}\right)\Delta\xi^2$. t^n is the time level of the known variables, while $t^{n+1} = t^n + \Delta t$ is the time level of the unknown variables. It must be stressed that the advancing in time of the solution is performed on the staggered grid. The state of the system is known in the center of the original grid and is defined by the cell averaged values $\tilde{\bar{h}}_{i;j}, \tilde{\bar{r}}_{i;j}$. The calculation procedure is based on the following sequence: 1) reconstruction of cell averages $\tilde{\bar{h}}^{(n)}_{i+\frac{1}{2};j+\frac{1}{2}}$ and $\tilde{\bar{r}}^{l(n)}_{i+\frac{1}{2};j+\frac{1}{2}}$ on the staggered grid from cell averages $\tilde{\bar{h}}^{(n)}_{i;j}, \tilde{\bar{r}}^{l(n)}_{i;j}$ on the original grid at time level t^n; 2) reconstruction of point values $h^{(n)}_{i;j}$ and $r^{l(n)}_{i;j}$ defined on the original grid from cell averages $\tilde{\bar{h}}^{(n)}_{i;j}$ and $\tilde{\bar{r}}^{l(n)}_{i;j}$ defined on original the grid at time level t^n; 3) reconstruction of flux derivatives on the original grid from point values $h^{(n)}_{i;j}$ and $r^{l(n)}_{i;j}$ on the original grid at time level t^n; 4) computation of point values $h^{(n+1/2)}_{i;j}$, $r^{l(n+1/2)}_{i;j}$, $h^{(n+1)}_{i;j}$ and $r^{l(n+1)}_{i;j}$ by a Natural Continuous Extension into a Runge-Kutta solver; 5) advancing from time level t^n to time level t^{n+1} of cell averages $\tilde{\bar{h}}^{(n+1)}_{i+\frac{1}{2};j+\frac{1}{2}}$ and $\tilde{\bar{r}}^{l(n+1)}_{i+\frac{1}{2};j+\frac{1}{2}}$ on staggered grid by the numerical integration of Equations (5) and (9); 6) destaggering by reconstruction of cell averages $\tilde{\bar{h}}^{(n+1)}_{i;j}$ and $\tilde{\bar{r}}^{l(n+1)}_{i;j}$ on the original grid from cell averages $\tilde{\bar{h}}^{(n+1)}_{i+\frac{1}{2};j+\frac{1}{2}}$ and $\tilde{\bar{r}}^{l(n+1)}_{i+\frac{1}{2};j+\frac{1}{2}}$ on the staggered grid.

Fourth-order accuracy in time is obtained by the calculation procedure indicated in Levy et al.[2] and Caleffi et al.[4] where a Simpson's quadrature rule is involved.

Time discretization of Equation (5) gives:

$$\tilde{\bar{h}}^{(n+1)}_{i+\frac{1}{2};j+\frac{1}{2}} = \tilde{\bar{h}}^{(n)}_{i+\frac{1}{2};j+\frac{1}{2}} - \frac{\Delta t}{\sqrt{\bar{g}}} \sum_{s=1}^{3} N_s \left[\sum_{\alpha=1}^{2} \left(\int_{\Delta\xi^{\alpha+}} \sqrt{g} r^\alpha d\xi^\beta - \int_{\Delta\xi^{\alpha-}} \sqrt{g} r^\alpha d\xi^\beta \right) \right]^{(t^n+\beta_s\Delta t)} \quad (10)$$

where $t^n + \beta_s \Delta t$ indicates the time level at which the term inside the square brackets is evaluated. $N_s = \left[\frac{1}{6}, \frac{2}{3}, \frac{1}{6}\right]^T$ are the weights and $\beta_s = \left[0, \frac{1}{2}, 1\right]$ are the nodes of the quadrature.

Time discretization of Equation (9) gives:

$$\tilde{\vec{r}}^{\,l(n+1)}_{i+\frac{1}{2};j+\frac{1}{2}} = \tilde{\vec{r}}^{\,l(n)}_{i+\frac{1}{2};j+\frac{1}{2}} - \frac{\Delta t}{\sqrt{\tilde{g}}} \sum_{s=1}^{3} N_s \Bigg\{ \int_{\Delta\xi^{\alpha+}} \tilde{\vec{g}}^{(l)} \cdot \vec{g}_{(k)} \frac{r^k r^\alpha}{h} \sqrt{g} d\xi^\beta - \int_{\Delta\xi^{\alpha-}} \tilde{\vec{g}}^{(l)} \cdot \vec{g}_{(k)} \frac{r^k r^\alpha}{h} \sqrt{g} d\xi^\beta$$
$$- \int_{\Delta\xi^1}\int_{\Delta\xi^2} \tilde{\vec{g}}^{(l)} \cdot \vec{g}_{(k)} \left[G h g^{mk} (h+H)_{,m} + R^l \right] \sqrt{g} d\xi^1 d\xi^2 \Bigg\}^{(t^n+\beta_s \Delta t)} \tag{11}$$

4 RESULTS AND DISCUSSION

In this section, the high-resolution shallow water equation solver described above is validated against two benchmark test cases: the circular dam break test and the symmetric channel with variable width test. The computed results are compared with analytical solutions and previously published predictions.

The circular dam break test, proposed by Alcrudo & Garcia-Navarro[7] and used by many authors (e.g. Liska & Wendroff[8]), consist of a cylindrical dam with radius $11m$ in the centre of a square domain $(0,\, 50\,m) \times (0,\, 50\,m)$. The initial water level is $10m$ inside the dam and $1m$ outside the dam and water is initially at rest. Suddenly, the cylindrical wall forming the dam is removed and time evolution of free surface and velocity fields are calculated. A first simulation (simulation S1) is carried out on a Cartesian grid made of 50×50 square cells, as in Alcrudo & Garcia-Navarro[7] and Liska & Wendroff[8]; a second simulation of the same test (simulation S2) is carried out on a curvilinear grid. In simulation S2 the number of calculation cells inside the square $(0,\, 50\,m) \times (0,\, 50\,m)$ is about 2425, that is similar to that used for the simulation carried out in Cartesian grid (2500). In order to compare our results with those obtained by Alcrudo & Garcia-Navarro[7] and Liska & Wendroff[8], Fig. 1 shows the contour plot of water level at $0.69s$. The numerical results obtained by our model on the grid used by Liska & Wendroff[8] (simulation S1) are shown in Fig. 1 left; the numerical results obtained by our model on the curvilinear grid (simulation S2) are shown in Fig. 1 right and in Fig. 3 left. As Fig. 1 shows, the circular symmetry is preserved very well; our numerical results are in good agreement with results obtained Liska & Wendroff[8].

A numerical simulation of a steady state flow with hydraulic and negative jumps is performed by reproducing the symmetric channel with variable width proposed in Liska & Wendroff[8]: for $x < 10m$, the channel width is equal to $40m$; for $10m < x < 30m$, the southern and northern channel wall inclines inward with an angle of $15°$ to the x direction; for $30m < x$ the channel width is constant and equal to $29.282m$. The initial and inflow conditions are the water depth $h_0 = 1m$ and Froude number $F_r = 2.5$. The simulation (simulation S3) is carried out on a deformed grid made of 72×32 cells (the same number of cells used by Liska & Wendroff[8]), obtained by imposing a concave-left curvature to the western and eastern boundaries of the domain used in Liska & Wendroff[8]. The grid used in simulation S3 is shown in Fig. 2 left. The numerical results obtained by our model in simulation S3 are shown in Fig. 2 right and in Fig. 3 right. Hydraulic jumps and negative

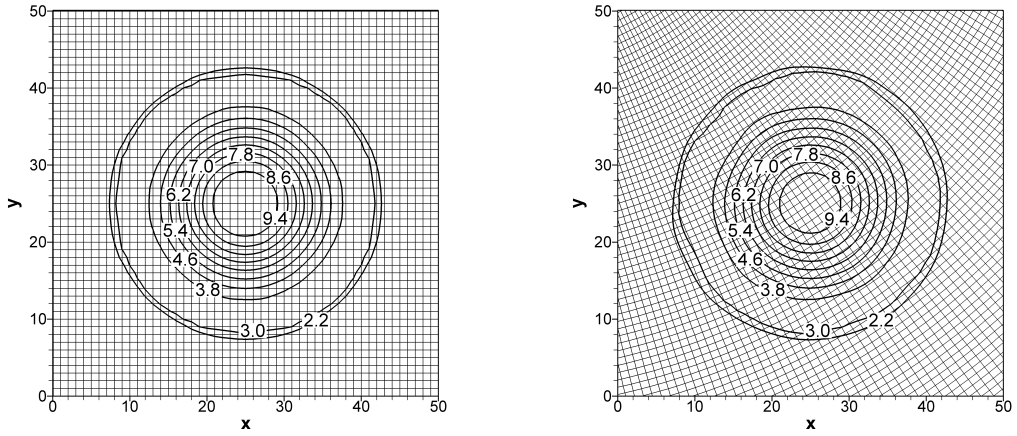

Figure 1: Circular dam break. Left: contour plot of the water depth at time 0.69s; results of the simulation carried out on the Cartesian grid used in Liska & Wendroff[8] (S1). Right: contour plot of the water depth at time 0.69s; results of the simulation carried out on the curvilinear grid (S2).

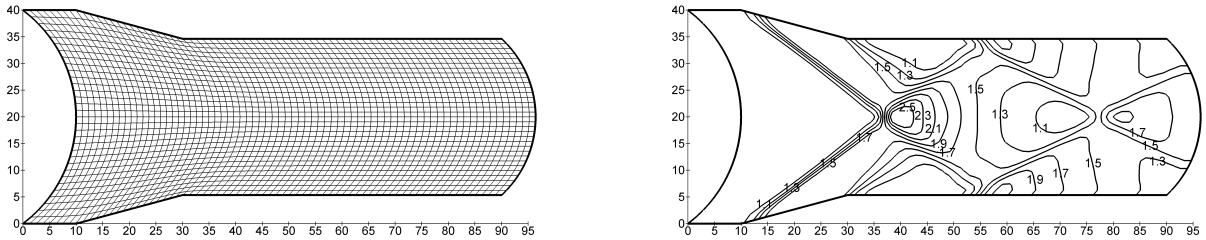

Figure 2: Symmetric channel with variable width. Left: calculation grid used in the simulation S3. Right: contour plot of the water depth (S3).

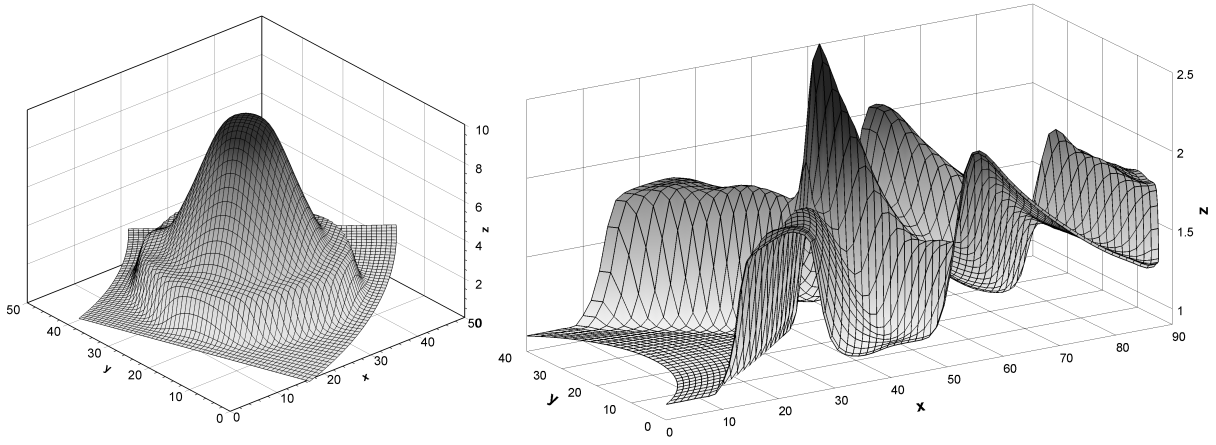

Figure 3: Left: circular dam break, surface plot of the water depth at time 0.69s; results of the simulation carried out on the curvilinear grid (S2). Right: symmetric channel with variable width, surface plot of the water depth (S3).

jumps caused by the presence of concave corners are resolved properly and the results are in good agreement with the numerical results presented by Liska & Wendroff[8].

5 CONCLUSIONS

A new Central Weighted Essentially Non-Oscillatory scheme for the solution of the shallow water equations expressed in contravariant formulation has been presented. A formal integral expression of the shallow water equations in contravariant formulation, in which Christoffel symbols are avoided, has been defined. The WENO reconstructions are performed by a two dimensional interpolating procedure taking into account the curved coordinate lines. The proposed scheme ensures the satisfaction of the exact C-property. The model is validated against two benchmark tests, and the results compare very well with theoretical and alternative numerical solutions.

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