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A PERTURBATIVE METHOD FOR DOUBLE-LAYER SHALLOW WATER EQUATIONS

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Summary. A double-layer formulation for 3D gravity currents is proposed in this paper. This formulation is obtained starting from the shallow water equations for two layers of immiscible liquids, with different densities and thicknesses, and then imposing the rigid-lid condition. As a consequence a non vanishing pressure p_s arises on the free surface of the upper layer and must be determined by solving a Poisson equation, together with momentum and mass balance equations. By means of a perturbative expansion of the field variables, the formulation of the problem is suitably simplified. The comparisons between numerical and experimental results are encouraging and show that numerical results are consistent with the experiments.

1 INTRODUCTION

Gravity currents are flows caused by density differences, both as the result of natural processes and of human activities. The book[1] and the recent review[2] provide a large variety of gravity current examples and an interesting synthesis of the main results that have been obtained on different types of gravity currents, while the book[3] gives an exhaustive review of their mathematical treatment. There is a huge sort of gravity currents. In this work we limit our interest to those consisting of two layers of immiscible liquids, with densities ρ_1, ρ_2 ($\rho_1 > \rho_2$) and thicknesses h_1, h_2. The lower layer, consisting of heavier liquid, is separated from the upper layer, consisting of lighter liquid, by a separation surface and the upper layer is limited by a free surface.

Actual gravity currents are mostly 3D, nevertheless they have not been investigated to the same extent as 2D and axisymmetric ones. Recent examples are represented by[4, 5, 6]. In the paper[4] the authors investigated a 3D gravity currents evolving on a uniform slope. This investigation was performed both experimentally and theoretically, by means of a box model.

A different box model was used in[5] to model the dynamics of a 3D gravity current, realized by means of a full-depth, lock-exchange release experiment. In[6] the same 3D gravity current of[5] was modeled by means of a single-layer, shallow water model, handled numerically with a finite-volume Godunov formulation[7] together with the use of the approximate Riemann solver of Roe[8].

The shallow water equations can be considered a good tool for the investigation of gravity currents[9]. On the contrary, the adoption of a single-layer formulation, obtained in the limit for $h_1+h_2\rightarrow\infty$, $h_1/(h_1+h_2)\rightarrow 0$, could be considered questionable when the orders of magnitude of h_1, h_2 are the same. The single-layer model should be applied when the values of h_2 are much larger than those of h_1, as in[10, 11], but it is also successfully applied when the orders of magnitude of h_1, h_2 are the same. Indeed, although in[12] are shown some weaknesses of the single-layer model in predicting the slumping behavior of 2D gravity currents, the works[13, 14] show how the single-layer model is a versatile tool for the prediction of thickness and velocity of 2D and axisymmetric gravity currents even for finite values of H ($H=h_1+h_2$) and for a wide range of the density ratios $\frac{\rho_2}{\rho_1}$.

In[6] the validity of the single-layer model has been shown for 3D gravity currents too. Nevertheless when gravity currents have $O(1)$ values of the ratio h_1/H, the motion of the upper layer is not negligible with respect to that of the lower layer. In this case, a double-layer formulation would be needed for modeling the motion of the upper layer. For 2D and axisymmetric gravity currents, the double-layer model has been applied successfully and its ability in reproducing correctly all of the phases of the gravity currents evolution has been shown in[12]. For 3D gravity currents, according to the authors' knowledge, the work is still in progress. So, the aim of this paper is to present some preliminary results concerned with the application of the shallow water, double-layer formulation to 3D gravity currents.

It is well known that such a formulation entails the definition of 4 partial differential equations, for 2D and axisymmetric gravity currents, and 6 partial differential equations for 3D gravity currents[15]. Anyway, imposing the rigid-lid condition (i.e. the free surface remains perfectly flat during the motion: $H=h_1+h_2=cost$) the number of equations decreases, because the evolution equation for the upper layer thickness is substituted by the algebraic relation: $H-h_1=h_2$. The disadvantage is that a non vanishing, unknown pressure distribution p_s must be considered on the free surface. But, unlike for 2D and axisymmetric gravity currents[12, 16], this pressure cannot be eliminated from the equations of motions and has to be determined solving them.

In this paper a computational strategy is proposed in order to solve such a problem. It is based on the fact that, as a consequence of the rigid-lid constraint, a Poisson equation is obtained for the pressure distribution p_s. This equation must be solved together with the equations of motion. However, by means of a suitable scaling and a perturbative expansion of the variables, the solution of the problem can be simplified.

The structure of the paper is as follows: first the mathematical model is formulated. Second, the computational strategy, aimed to determine the pressure distribution p_s on the free surface and to solve the equations of motion is defined. Third, a suitable scaling of the variables and a perturbative expansion are introduced in order to simplify the mathematical formulation. Fourth, some preliminary numerical results are compared with experiments realized with the experimental setup described in[6] and discussed.

The comparisons are found encouraging. Future work is addressed to a systematic validation of the mathematical model.

2 THE MATHEMATICAL MODEL

Consider two layers of incompressible, immiscible liquids, heights h_1, h_2, densities ρ_1, ρ_2 ($\rho_1>\rho_2$), evolving in a domain with horizontal characteristic dimension L. Assuming that the parameter σ (defined as: $\sigma=H/L$) is small ($\sigma<<1$), it is possible to show[15] that the hydrostatic pressure distribution is obtained from the vertical component of the momentum equation of each layer, at the leading order, having adopted σ as the small parameter of a perturbative expansion. Then, assuming that the pressure value on the free surface ($z=H=h_1+h_2$) is equal to p_s, the pressure distributions within the two layers are given by:

$$p_1 = p_s + \rho_1 g(h_1 - z) + \rho_2 g h_2,\; p_2 = p_s + \rho_2 g(h_1 + h_2 - z) \tag{1}$$

Substituting the pressure distributions (1) in the horizontal components of the momentum equation of each layer, averaging them with respect to the vertical coordinate, and accounting for suitable kinematic conditions resulting from the immiscibility hypothesis (not reported for the sake of simplicity), the following double-layer model is obtained[15] :

$$\begin{aligned}
&\frac{\partial h_1}{\partial t}+\frac{\partial p_1}{\partial x}+\frac{\partial q_1}{\partial y}=0\\
&\frac{\partial p_1}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p_1^2}{h_1}+g\frac{h_1^2}{2}\right)+g\frac{\rho_2}{\rho_1}h_1\frac{\partial h_2}{\partial x}+\frac{\partial}{\partial y}\left(\frac{p_1 q_1}{h_1}\right)=-\frac{h_1}{\rho_1}\frac{\partial p_s}{\partial x}-\frac{1}{\rho_1}T_{xz}^1\Big|_{z=0}\\
&\frac{\partial q_1}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p_1 q_1}{h_1}\right)+\frac{\partial}{\partial y}\left(\frac{q_1^2}{h_1}+g\frac{h_1^2}{2}\right)+g\frac{\rho_2}{\rho_1}h_1\frac{\partial h_2}{\partial y}=-\frac{h_1}{\rho_1}\frac{\partial p_s}{\partial y}-\frac{1}{\rho_1}T_{yz}^1\Big|_{z=0}\;,\\
&\frac{\partial h_2}{\partial t}+\frac{\partial p_2}{\partial x}+\frac{\partial q_2}{\partial y}=0\\
&\frac{\partial p_2}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p_2^2}{h_2}+g\frac{h_2^2}{2}\right)+gh_2\frac{\partial h_1}{\partial x}+\frac{\partial}{\partial y}\left(\frac{p_2 q_2}{h_2}\right)=-\frac{h_2}{\rho_2}\frac{\partial p_s}{\partial x}\\
&\frac{\partial q_2}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p_2 q_2}{h_2}\right)+\frac{\partial}{\partial y}\left(\frac{q_2^2}{h_2}+g\frac{h_2^2}{2}\right)+gh_2\frac{\partial h_1}{\partial y}=-\frac{h_2}{\rho_2}\frac{\partial p_s}{\partial y}
\end{aligned}\tag{2}$$

where p_1, q_1, p_2, q_2 are the volumetric fluxes per unit length, along x and y directions respectively, defined as the product of the vertically average velocity components (u_1, v_1, u_2, v_2) by the correspondent layer's height (h_1, h_2). Interfacial stresses are neglected and the bottom stress $T_{xz}^1\Big|_{z=0}, T_{yz}\Big|_{z=0}$ are calculated by means of a friction coefficient, as in[6]. The pressure p_s is equal to zero if the upper layer is limited by a free surface, as usual. In this case the components of the gradient of the pressure p_s disappear from equations (2).

Consider the sum of first and fourth equation (2):

$$\frac{\partial}{\partial t}(h_1 + h_2) + \frac{\partial}{\partial x}(p_1 + p_2) + \frac{\partial}{\partial y}(q_1 + q_2) = 0 \tag{3}$$

and assume that H is constant with respect to both time and space. It follows from (3) that:

$$\frac{\partial}{\partial x}(p_1 + p_2) + \frac{\partial}{\partial y}(q_1 + q_2) = 0 \tag{4}$$

i.e. that the vectorial field $\mathbf{V} \equiv (p_1 + p_1, q_1 + q_2)$ is divergence-free. Moreover, the evolutive equation for h_2 can be discarded and substituted by the algebraic relation: $H = h_1 + h_2$. The number of equations of the partial differential system (2) is so reduced by one. The reduced partial differential system is defined straightforwardly and is not reported for the sake of simplicity. The main disadvantage is represented by the fact that the pressure p_s on the free surface, being forced to remain flat, is no longer zero but becomes an unknown of the problem and must be determined together with the equations of motion.

The equation for p_s is obtained from equations (2) as follows. First, (neglecting bottom stresses for the sake of simplicity) consider the sum of second and fifth equation (2) and third and sixth equation (2):

$$\begin{aligned}
&\frac{\partial U}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p_1^2}{h_1} + \frac{p_2^2}{h_2} + g'\frac{h_1^2}{2}\right) + \frac{\partial}{\partial y}\left(\frac{p_1 q_1}{h_1} + \frac{p_2 q_2}{h_2}\right) = -a\frac{\partial p_s}{\partial x} \\
&\frac{\partial V}{\partial t} + \frac{\partial}{\partial x}\left(\frac{p_1 q_1}{h_1} + \frac{p_2 q_2}{h_2}\right) + \frac{\partial}{\partial y}\left(\frac{q_1^2}{h_1} + \frac{q_2^2}{h_2} + g'\frac{h_1^2}{2}\right) = -a\frac{\partial p_s}{\partial y}
\end{aligned} \tag{5}$$

being: $U \equiv p_1 + p_2, V \equiv q_1 + q_2, a \equiv \left(\frac{h_1}{\rho_1} + \frac{h_2}{\rho_2}\right)$. Second, differentiate first equation (5) with respect to x, second equation (5) with respect to y and sum them. Third, invoking the divergence-free condition (4), the following Poisson equation is obtained for p_s:

$$\frac{\partial^2}{\partial x^2}\left(\frac{p_1^2}{h_1} + \frac{p_2^2}{h_2} + g'\frac{h_1^2}{2}\right) + 2\frac{\partial^2}{\partial y \partial x}\left(\frac{p_1 q_1}{h_1} + \frac{p_2 q_2}{h_2}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{q_1^2}{h_1} + \frac{q_2^2}{h_2} + g'\frac{h_1^2}{2}\right) = -\frac{\partial}{\partial x}\left(a\frac{\partial p_s}{\partial x}\right) - \frac{\partial}{\partial y}\left(a\frac{\partial p_s}{\partial y}\right) \tag{6}$$

Solving together the reduced partial differential system and the equation (6) the unknowns $h_1, p_1, q_1, p_2, q_2, p_s$ can be determined. Initial and boundary conditions are determined by the motion condition under consideration.

3 SCALING AND PERTURBATIVE EXPANSIONS OF THE VARIABLES

Consider the following scaling of the variables:

$$\begin{aligned}
&h_1 = h\widetilde{h}_1, h_2 = H\left(1 - \frac{h}{H}\right)\widetilde{h}_2,\ p_1 = \sqrt{g'h}h\,\widetilde{p}_1, q_1 = \sqrt{g'h}h\,\widetilde{q}_1,\ p_2 = \sqrt{g'h}H\left(1 - \frac{h}{H}\right)\widetilde{p}_2, q_2 = \sqrt{g'h}H\left(1 - \frac{h}{H}\right)\widetilde{q}_2, \\
&T_{xz}\big|_{z=0} = \rho f_a g' h \widetilde{T}_{xz}, T_{yz}\big|_{z=0} = \rho f_a g' h \widetilde{T}_{yz}, t = L\big/\sqrt{g'h}\,\widetilde{t},\ x = L\,\widetilde{x},\ y = L\,\widetilde{y},\ p_s = \frac{h}{H}\rho g' h\,\widetilde{p}_s
\end{aligned} \tag{7}$$

where h is the order of magnitude of the lower layer thickness, L is the order of magnitude of a characteristic horizontal dimension, f_a is the bottom friction coefficient and ρ is defined as the mean value of ρ_1, ρ_2: $\rho=(\rho_1+\rho_2)/2$. The ratio h/H is assumed to be small enough ($h/H<1$) and defined as: $\varepsilon = h/H$.

The reduced partial differential system assumes the following dimensionless form (for simplicity of notation tilde is omitted from dimensionless quantities):

$$
\begin{aligned}
&\frac{\partial h_1}{\partial t}+\frac{\partial p_1}{\partial x}+\frac{\partial q_1}{\partial y}=0\\
&\frac{\partial p_1}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p_1^2}{h_1}+\frac{h_1^2}{2}\right)+\frac{\partial}{\partial y}\left(\frac{p_1 q_1}{h_1}\right)=-\varepsilon\frac{\rho}{\rho_1}h_1\frac{\partial p_s}{\partial x}-\left(f_a\frac{\rho}{\rho_1}\frac{L}{h}\right)T_{xz}\\
&\frac{\partial q_1}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p_1 q_1}{h_1}\right)+\frac{\partial}{\partial y}\left(\frac{q_1^2}{h_1}+\frac{h_1^2}{2}\right)=-\varepsilon\frac{\rho}{\rho_1}h_1\frac{\partial p_s}{\partial y}-\left(f_a\frac{\rho}{\rho_1}\frac{L}{h}\right)T_{yz}\\
&\frac{\partial p_2}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p_2^2}{h_2}\right)+\frac{\partial}{\partial y}\left(\frac{p_2 q_2}{h_2}\right)=-\varepsilon\frac{\rho}{\rho_2}h_2\frac{\partial p_s}{\partial x}\\
&\frac{\partial q_2}{\partial t}+\frac{\partial}{\partial x}\left(\frac{p_2 q_2}{h_2}\right)+\frac{\partial}{\partial y}\left(\frac{q_2^2}{h_2}\right)=-\varepsilon\frac{\rho}{\rho_2}h_2\frac{\partial p_s}{\partial y}
\end{aligned}
\tag{8}
$$

The adopted scaling gives a consistent result in the limit $\varepsilon\to 0$. In this case, as for 2D and axisymmetric gravity currents[12], the terms containing the pressure p_s vanish and the differential system (8) tends to the single-layer differential system.

Assume the following first-order, perturbative expansions:

$$
h_1=h_1^0+\varepsilon h_1^1,\ p_1=p_1^0+\varepsilon p_1^1,\ q_1=q_1^0+\varepsilon q_1^1,\ p_2=p_2^0+\varepsilon p_2^1,\ q_2=q_2^0+\varepsilon q_2^1 \tag{9}
$$

and substitute them into the differential system (8). This latter is split into an ε^0 order partial differential system, coinciding with the single-layer partial differential system[6] and governing the evolution of the ε^0 order quantities: $h_1^0, p_1^0, q_1^0, p_2^0, q_2^0$, and into an ε^1 order partial differential system, containing the pressure p_s and governing the evolution of the ε^1 order quantities: $h_1^1, p_1^1, q_1^1, p_2^1, q_2^1$. Both of them are not reported for the sake of simplicity. To solve the ε^1 order partial differential system, it is necessary to know the ε^0 order quantities: $h_1^0, p_1^0, q_1^0, p_2^0, q_2^0$ and the pressure p_s. This latter is obtained solving equation (6), scaled according to (7) and accounting for the expansions (9). At the leading order and considering Boussinesq's gravity currents ($\frac{\rho_2}{\rho_1}\approx 1$), equation (6) becomes:

$$
\frac{\partial^2 p_s}{\partial x^2}+\frac{\partial^2 p_s}{\partial y^2}=-\frac{\partial^2}{\partial x^2}\left(\frac{\left(p_1^0\right)^2}{h_1^0}+\frac{\left(h_1^0\right)^2}{2}\right)-2\frac{\partial^2}{\partial y\partial x}\left(\frac{p_1^0 q_1^0}{h_1^0}\right)-\frac{\partial^2}{\partial y^2}\left(\frac{\left(q_1^0\right)^2}{h_1^0}+\frac{\left(h_1^0\right)^2}{2}\right) \tag{10}
$$

The ε^0 order partial differential system, the Poisson equation (10) for the pressure and the ε^1 order partial differential system are solved sequentially. The complete solution is then expressed by (9).

4 RESULTS

The ε^0 order partial differential system, the Poisson equation (9) for the pressure and the ε^1 order partial differential system are solved by means of the finite volume method described in[6], the standard SOR method and the Lax Wendroff methods respectively. Numerical simulations are aimed to reproduce gravity currents realized with the experimental setup described in[6]. It is a full-depth, lock-exchange release experiment realized with two identical volumes of liquid, (square section parallelepipeds, height H, side L=1 m), densities ρ_1 (salty water), ρ_2 (fresh water ρ_2≈1000 kg/mc), separated by a lock with a width b (b=0.2 m). As soon as the lock is manually lifted, the gravity current starts its motion. In figure 1, the top view of an experimental gravity current is shown: it has the typical mushroom like-shape described in[6]. The complex structure of the lobe and cleft's instability[1] is clearly visible along the contour of the expanding front, while, although not distinguishable in figure 1, a complex vortex, similar to that described in[17] for axisymmetric gravity currents, has been observed.

A quantitative comparison between numerical and experimental data is shown in figure 2a,b. It shows (figure 2a) the dimensionless position of the front x_f* along x axis ($x_f^* = x_f/L$) (see figure 1) versus dimensionless time t* ($t^* = t\sqrt{g'h}/L$) and the top view of the gravity current at different instants of time (figure 2b).

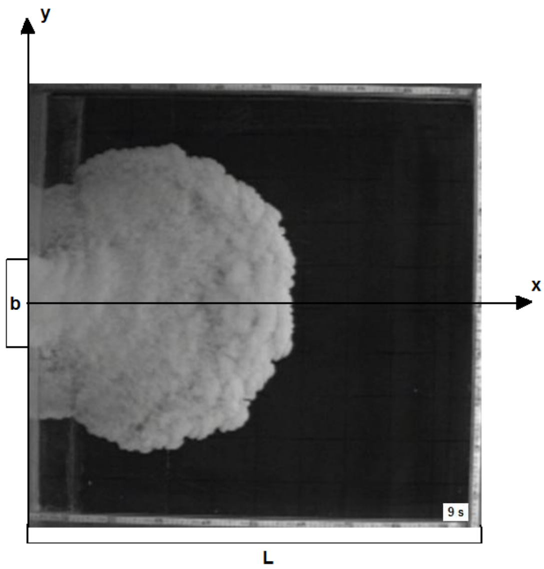

Figure 1. Top view of an experimental 3D gravity current. H=0.1 m, ρ_1=1018 kg/mc, ρ_2=1000 kg/mc, 9 s after the start of the experiment. b=0.2 m, L=1 m.

Plots in figure 2b are made with dimensional quantities. Data in figure 2a have been obtained with: H=0.2 m, ρ_1=1018 kg/mc, ρ_1=1033 kg/mc, e_s=0 mm, e_s=3 mm. e_s is the bottom roughness (see the work[6])

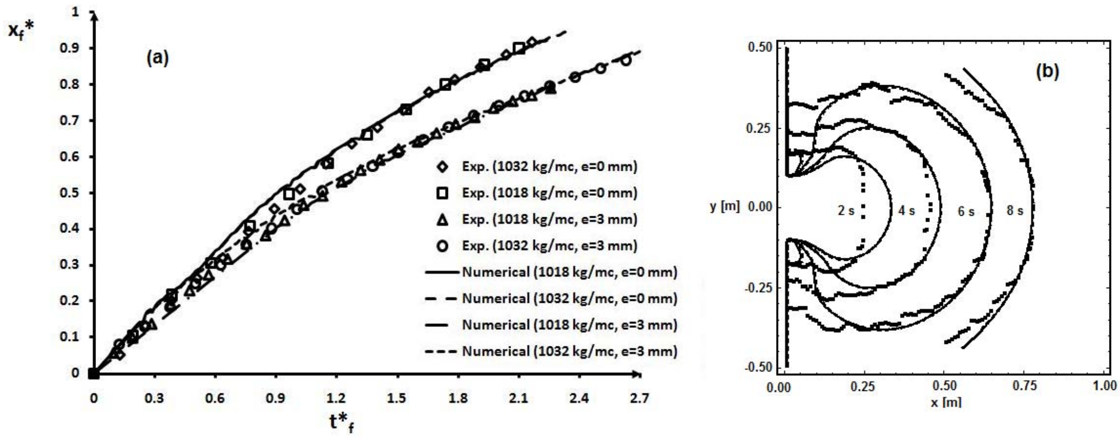

Figure 2a. Dimensionless front position versus dimensionless time. Figure 2b. Top view of profile of the gravity current's front. H=0.2 m, ρ_1=1032 kg/mc, ρ_2=1000 kg/mc.

In figure 2a, the results relative to different densities and same roughness condition collapse on the same curve. The two curves separate from each other when the roughness, i.e. the bottom friction, starts to affect the motion. Experimental and numerical data are in fairly good agreement.

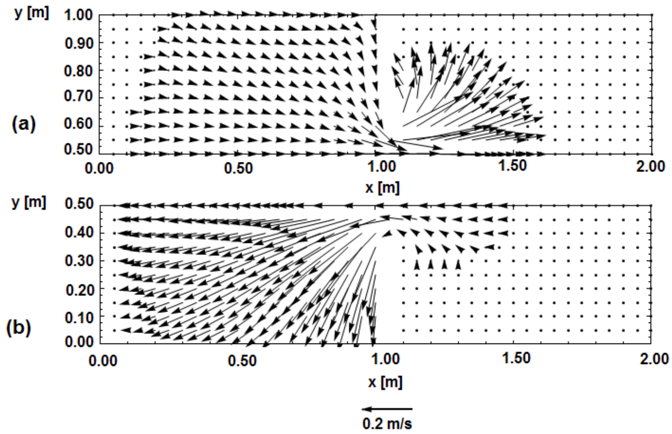

Figure 3. Velocity fields of the gravity current (a) and the upper layer (b). H=0.15 m, , ρ_1=1018 kg/mc, ρ_2=1000 kg/mc. 6 s after the start of the run.

In figure 3 the numerical velocity fields of the upper and the lower layer are plotted. Although no experimental results have been obtained for the velocity fields, the numerical results appear qualitatively consistent with the experimental observations.

5 CONCLUSIONS

A double-layer model for 3D gravity currents has been proposed in this paper. By means of a suitable scaling and a perturbative expansion of the variables, the problem has been split into three sub-problems which have to be solved sequentially. Results are encouraging, although the method needs an exhaustive validation, which will be matter for future work.

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