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HOMOGENIZATION AND UPSCALING OF FLOW THROUGH VEGETATION

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Summary. Vegetated marshes and wetlands are complex hydrological systems. Due primarily to computational constraints, the effects of vegetation must be incorporated into depth-integrated flow models through empirical or theoretical resistance parameterizations that are closely related to the classical Chezy, Manning, and Darcy laws for surface roughness and porous media. In this work we investigate the use of homogenization techniques and multi-scale numerical modeling to represent the effect of vegetation on shallow water flow. In particular, we use highly resolved two- and three-dimensional numerical models of flow through computer-generated vegetated, flow domains along with modern volume averaging and homogenization techniques to better understand flow resistance in various flow regimes. The numerical models use unstructured meshes to resolve complex geometry, and the variational multi-scale (finite element) method to resolve steady and unsteady, low and high Reynolds number flows.

1 INTRODUCTION

Accurate predictive models of shallow water flows through coastal environments are needed for a number of industrial, military, civil works and emergency management applications. In many of these applications, for example coastal inundation due to hurricane storm surges, large domains which extend beyond the region of interest are required to capture the basin to coastal shelf to overland flow characteristics. Modeling coastal flow processes is complicated by the presence of smaller-scale features, including stationary and moving man-made structures, vegetated marshlands, barrier islands, etc. Current mathematical models and parameterizations of flow resistance characteristics due to, for example, flow over and around porous structures or through heavily vegetated regions are inadequate to accurately capture these physical processes and, as a result, numerical models based on these mathematical representations may be inaccurate. Furthermore, while the use of highly resolved, 3D Navier-Stokes models may avoid some of the parameterization issues, the implementation of such models over the large computational domains of interest is prohibitive. An alternative is to appropriately couple mathematical/numerical models that accurately and efficiently capture the primary flow characteristics of a given region within the larger flow domain.

In this report we focus on the behavior of flow of water through a vegetated region. The existence of vegetation affects the flow resistance, which is a major factor in determining velocity and pressure distribution. The main impact of vegetation is the withdrawal of momentum from the flow.[2] Since modeling the effects on flow from each individual plant is impractical in a large-scale simulation, where computational resolution is on the order of tens of meters, we focus on methods for describing this sub-grid scale phenomena. One possible approach to deal with the effects of form drag from vegetation is upscaling through the process of homogenization. This is a common technique in studying flow through porous media, which shares many of the characteristics of the problem at hand. If the geometry of the vegetated domain has a certain degree of periodicity, then by analyzing the effect of vegetation on a small section of the domain, we can upscale to the entire domain.

Another way we address the effect of vegetation on flow is to look at bottom roughness in channel flow. Parameterizations like Manning's Law and Chezy's Law include a drag coefficient which represents the roughness of the channel's bottom. By treating vegetation as another form of bottom roughness we attempt to upscale their effects into this type of law. The result depends greatly on the Reynolds number of the flow.

2 THEORY

2.1 HOMOGENIZATION

One approach to modeling flow through densely vegetated regions is to treat the region as a porous medium, where the vegetation represents the rock matrix and the space between plants is the pore space. Under this assumption, the flow can be upscaled using the method of homogenization through a periodic porous medium. In this method, we consider a family of functions u^ϵ where $\epsilon > 0$ is a spatial scale parameter, and determine the limit $u = \lim_{\epsilon \to 0} u^\epsilon$. The result is an upscaled solution. We calculate each function u^ϵ by solving a periodic cell problem.

For simplicity, we assume that we have a periodic perforated domain. Let Y be a standard periodicity cell with a standard obstacle $G \subset\subset Y$, representing the vegetation, with a piecewise smooth boundary Γ. The remainder is denoted $B = Y - G$. This is the "fluid region". We assume the standard geometry of Y is repeated throughout $\mathbf{R^N}$.

Now, we perform homogenization on the standard Stokes Problem to formally derive Darcy's Law. We start with the following problem on the pore scale

$$\begin{aligned} \epsilon^2\mu\Delta v^\epsilon(x) = \nabla p^\epsilon(x), &\quad x \in B^\epsilon, \\ \nabla\cdot v^\epsilon(x) = 0 &\quad x \in B^\epsilon \\ v^\epsilon(x) = 0, &\quad x \in \Gamma^\epsilon \end{aligned} \tag{1}$$

where μ is the dynamic viscosity, v is velocity and p is pressure.

Now the cell problem is to find the Y-periodic vector fields $w_j(y)$ with components $w_{ji}(y)$ that solve the Stokes problems,

$$\begin{aligned} \nu\Delta_y w_j(y) = \nabla_y \tfrac{\pi_j(y)}{\rho} - e_j &\quad y \in B \\ \nabla\cdot w_j(y) = 0 &\quad y \in B \\ w_j(y) = 0 &\quad y \in \Gamma \end{aligned} \tag{2}$$

where $\pi_j(y)$ are the corresponding Y-periodic pressure fields, ν is the kinematic viscosity and ρ is the fluid density. Notice that the third equation is our standard no-slip condition on the walls of the vegetation.

After solving these Stokes problems we can calculate

$$k_{ij} = \nu \int_B w_{ji}(y)dy. \tag{3}$$

By introducing the tensor $K = k_{ij}$ and realizing that our result is divergence free, we have

$$V(x) = -\frac{1}{\mu}K\nabla p(x), \quad \nabla\cdot V = 0, \tag{4}$$

where K is symmetric and positive definite and V is the average velocity. Thus V and p satisfy a form of Darcy's Law.[5]

For flow in porous media we define $Re = \frac{Vd}{\nu}$ where V is average velocity, d is average particle diameter, and ν is kinematic viscosity. For $Re > 1$ we see that Darcy's Law no longer is valid because the inertial forces have overtaken the viscous forces. In this range, nonlinear behavior is seen. The inertial terms in Navier-Stokes are no longer negligible, so the homogenization of Stokes equation is no longer the correct upscaling. It has long been noted that for higher Reynolds numbers, the relationship looks like Darcy's Law plus a quadratic term. This is called the Darcy-Forchheimer equation

$$-\nabla p = \frac{\mu}{K}V + \beta\rho V^2. \tag{5}$$

Some evidence suggests that this additional term should be cubic.[1]

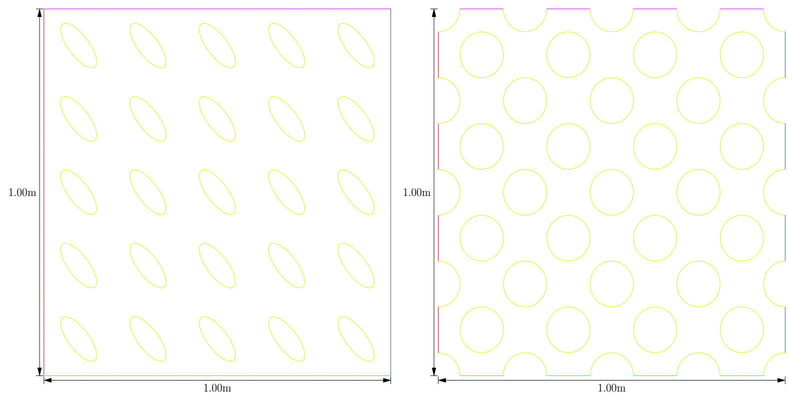

Figure 1: Examples of Domains used in Homogenization

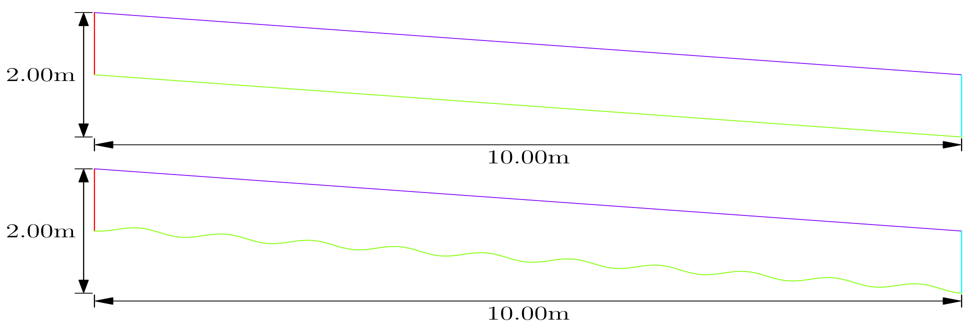

Figure 2: Examples of Domains used in Rough Channel Simulations

2.2 ROUGH CHANNEL FLOW

From empirical results in channel flow, it is common in hydraulics to assume

$$V = -\frac{(u-z)^{\alpha-1}}{c_f}\frac{\nabla u}{|\nabla u|^{1-\gamma}} \tag{6}$$

where $h(x,t) = u(x,t) - z(x)$ is the water depth, $u(x,t)$ is the free water surface elevation, $z(x)$ is the bed surface bathymetry, $V(x,t)$ is the depth-averaged velocity, and α and γ are constants with $1 < \alpha < 2$ and $0 < \gamma < 1$. For $\alpha = 5/3$ and $\gamma = 1/2$ we have Manning's equation, and for $\alpha = 3/2$ and $\gamma = 1/2$ we have Chezy's equation. These equations have been used for decades and were originally based on empirical observations; however, there do exist physical derivations of them. Manning's Equation has been derived using the asymptotic behavior of flow of incomplete similarity in the relative roughness.[4] Chezy's Equation can be derived by looking at the balance between the frictional force of the river bed and the gravitational force.[3] This comes from the assumption that frictional force is

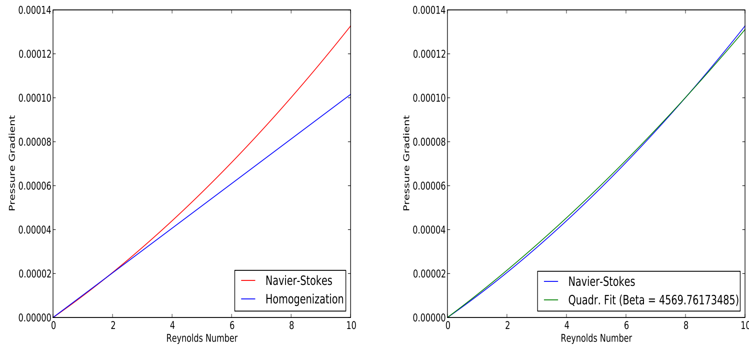

Figure 3: Left figure: A comparison of results from homogenization and from solving the full Navier-Stokes Equations for a range of Reynolds Numbers for packing of circles. Right figure: A comparison of results from solving the full Navier-Stokes Equations and our result from homogenization including the quadratic term from Darcy-Forchheimer on the same domain.

proportional to V^2. We desire to computationally calculate α and γ in order to justify this type of equation.

From empirical evidence, a Manning or Chezy type equation is generally valid for fully turbulent flow. However, it does not accurately upscale laminar flow or flow in the critical region between laminar flow and fully turbulent flow. For low Reynolds Numbers empirical results suggest that there should be a linear relationship with respect to slope

$$V = -\frac{1}{c_f}|\nabla h|. \tag{7}$$

As the Reynolds number increases this should become nonlinear and eventually will become turbulent.[7]

3 NUMERICAL METHOD AND BOUNDARY CONDITIONS

3.1 HOMOGENIZATION

We use a locally conservative, stabilized finite element method to solve the Two-Dimensional Stokes problem (2).[6] We assume periodic boundary conditions to ensure a divergence free velocity. We calculate the tensor K from Equation 3. This gives us the upscaled result in Equation 4. Homogenizations were performed on a variety of perforated 2D domains, including packings of circles and ellipses in various patterns.

Next, using Darcy's Law and our permeability tensor K, we find the average velocity over the domain. We compare our results to solving the full Navier-Stokes equations including the inertial terms to steady state on the same domain.

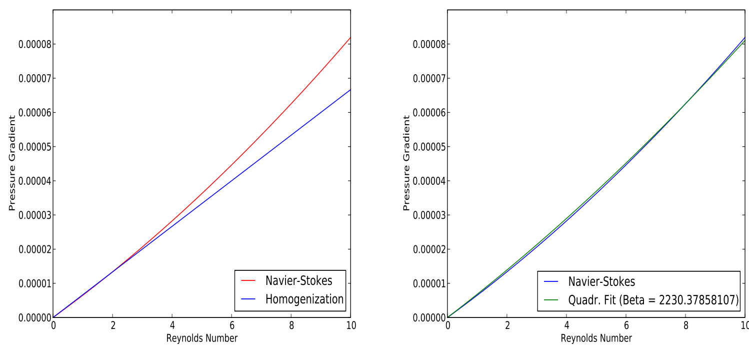

Figure 4: Left figure: A comparison of results from homogenization and from solving the full Navier-Stokes Equations for a range of Reynolds Numbers for packing of ellipses. Right figure: A comparison of results from solving the full Navier-Stokes Equations and our result from homogenization including the quadratic term from Darcy-Forchheimer on the same domain.

3.2 ROUGH CHANNEL FLOW

We want to verify a Manning-type equation by solving 2D Navier-Stokes Equations on sloped domains. We solve the full Navier-Stokes equations on various sloped domains with different bottom types, including flat-bottomed, sinusoidal-bottomed, and square wave-bottomed. Examples of these are seen in Figure 2. For boundary conditions, we need to use a no slip condition for velocity on the channel bottom and a slip condition on the surface. To simulate a long channel, we impose periodic velocity conditions on the inflow and outflow boundaries. We want pressure to be hydrostatic with respect to water depth, so we impose the hydrostatic condition on the inflow and outflow boundaries and use hydrostatic pressure as the initial condition for pressure. The flow in the direction of the slope is driven entirely by the gravitational term since there is no pressure difference between the inflow and outflow at a given depth. After solving the equations on these domains until it reaches steady state, we volume average to find the depth-averaged velocity.

4 RESULTS

4.1 HOMOGENIZATION

We implemented our method on several square 2D domains containing different packs of spheres and ellipses of varying orientation. After solving the Stokes' problems for the given domain, we used Darcy's Law to calculate the average velocity. Then, on the same domain, we solve the Navier-Stokes equations with periodic boundary conditions until it reaches steady state. Then, we volume average to find the average velocity. We do this over a range of many pressure gradients in order to capture the nonlinear behavior in flows with higher Reynolds numbers.

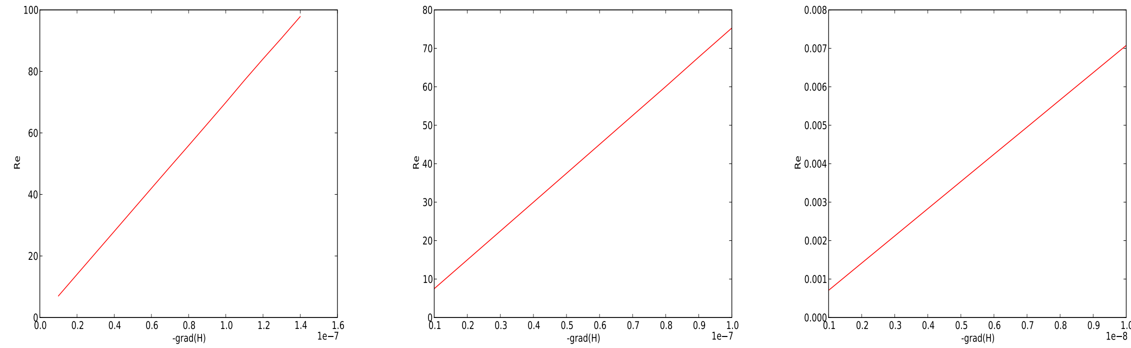

Figure 5: Reynolds numbers over a range of slopes for flat-bottomed, sinusoidal-bottomed, and square-wave-bottomed 2D domains respectively.

Over many arrangements of domains, we compared these two results, and found that for all of them, the results from homogenization were extremely close to the results from the full Navier-Stokes for small Reynolds number (< 1) as the theory implies should happen. In general, when using highly refined meshes for flows with Reynolds Number less than 1, the average velocity from homogenization was within 3 percent of the velocity from solving the full Navier-Stokes Equations. Figures 3 and 4 show the comparison between the homogenized result and the results full Navier-Stokes Equations over a range of pressure gradients for a packing of circles and a packing of ellipses, respectively.

Notice that in Figures 3 and 4 that for higher Reynolds numbers, we observe the expected nonlinear behavior. This suggests that the Darcy-Forchheimer Law is in effect. We need to calculate the quadratic coefficient β. From the results from the full Navier-Stokes, we subtract the linear part predicted by Darcy's Law by homogenization. Then we perform a least squares fit of the result to find the best quadratic coefficient β. Notice that in Figures 3 and 4 that our results for Darcy-Forchheimer match very closely to that from the full Navier-Stokes over a large range of Reynolds numbers. This suggests that the form of the Darcy-Forchheimer Law that we have used is generally accurate for this range of Reynolds numbers, between 0 and 10. Our method for calculating the Darcy-Forchheimer coefficients generally has error less than 10 percent over this range of Reynolds numbers, far outperforming those calculated by formulas created by Ergun and Kadlec and Knight.[3]

4.2 ROUGH CHANNEL FLOW

We implemented our method over several types of two-dimensional sloped domains. These were, flat-bottomed, sinusoidal-bottomed, and square wave-bottomed. Examples can be seen in Figure 2. We calculate the average velocity over a range of slopes. As seen in Figure 5, over a range of small slopes we see a linear relationship between mean

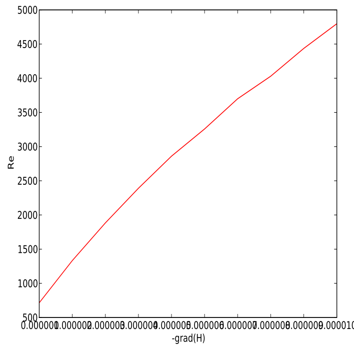

Figure 6: Reynolds numbers over a range of slopes for a sinusoidal-bottomed 2D domain. Notice the nonlinearity.

velocity and slope. This verifies empirical results and is what we expect from theory. As the slope is increased, we begin to see nonlinear behavior. In Figure 6, we see nonlinearity in the relationship for a sinusoidal-bottomed domain. This flow is not fully turbulent, so Manning or Chezy's Law is not yet valid.

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