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A HYBRID LEVEL-SET METHOD FOR FREE-SURFACE FLOWS

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Summary. Two-phase Navier-Stokes models are becoming increasingly popular for modeling free-surface flows and hydrodynamic processes. They hold particular appeal for problems where full vertical resolution is required in velocity and pressure, (e.g., short-wave phenomena, flow around coastal structures and levees, and extreme erosion processes). Level-set and volume-of-fluid formulations are the two most common approaches for modeling two-phase flows and both can be used across many flow regimes. Both share an advantage over front-tracking methods in that they are robust through changes in connectivity of the phases such as during bubble formation and wave breaking.

However, standard level-set methods do not conserve mass. The conservation errors are the result of describing interface dynamics using a level-set formulation and are not specific to the discrete approximation. Since conservation errors accumulate to produce qualitatively incorrect solutions, several researchers have attempted to address this issue by using hybrid level-set/volume-of-fluid and hybrid level-set/particle-tracking approaches. In this work we present a method for correcting the level set in order to control mass conservation error. The correction is defined as the solution of a nonlinear reaction-diffusion equation and can be applied to higher order finite element methods on unstructured meshes. Numerical results are presented for linear and quadratic approximations of incompressible air/water flows.

1 INTRODUCTION

Traditionally, depth-integrated formulations have been preferred for modeling free-surface hydrodynamics in the water resources community[11,17]. However, fully three-dimensional air/water flow is an essential feature of many important phenomena like wave breaking, overtopping of coastal structures, rapid erosion processes, as well as wave/current/vessel interaction. Depth-integrated models like the shallow water or Boussinesq equations may be inadequate in these settings because they are unable to reproduce the three-dimensional effects, phase interactions, and/or topological changes of the air/water interface[4,1]. For this class of problems, two-phase models based on the full Navier-Stokes equations have become increasingly popular[9,8].

There are, in fact, a number of approaches capable of approximating the full three-dimensional behavior of the two fluid system, including moving mesh methods, volume-of-fluid methods, level-set methods, and particle methods (see for example[20,16,15,13]). The approach developed here is based on level-set and volume-of-fluid methods because they are robust through topological changes in the fluid distributions while still applicable for high Reynolds number flow in large-scale, three-dimensional domains.

Level-set methods are our point of departure. Among other things, they deal well with topological changes in the free surface, are easy to implement in two or three-dimensions, and yield geometric information about interfaces directly[16]. However, unlike volume-of-fluid methods, they are not strictly mass (or volume) conserving[18]. To be more concrete, level-set methods describe the boundary between the two fluids implicitly as a zero level set of a scalar function defined in the problem domain[19]. For two viscous fluids in a domain Ω separated by a sharp interface Γ, the level-set description of the interface motion can be written as

$$\frac{\partial \phi}{\partial t} + \boldsymbol{u} \cdot \nabla \phi = 0 \tag{1}$$

where $\boldsymbol{u}$ is the velocity, and Γ is defined implicitly by the zero level set of ϕ

$$\Gamma = \{\boldsymbol{x} \mid \phi(\boldsymbol{x}) = 0\} \tag{2}$$

This equation is derived based on kinematics at Γ and the (non-unique) extension of the equation along Γ to all of Ω. Let one fluid have density ρ_a and occupy the domain $\Omega_a(t) = \{\mathbf{x} | \phi(\mathbf{x}, t) > 0\}$. The mass of the fluid is given by

$$M_a = \int_{\Omega_a(t)} \rho_a dV = \int_{\Omega} \rho_a H(\phi) dV \tag{3}$$

where H is the Heaviside function

$$H(\phi) = \begin{cases} 0 & \phi < 0 \\ 1/2 & \phi = 0 \\ 1 & \phi > 0 \end{cases} \tag{4}$$

Note that $H(\phi(x,t))$ is in this case the characteristic function of the set $\Omega_a(t)$. Conservation of mass over the time interval $[t_n, t_{n+1}]$ is then

$$\int_{\Omega} \rho_a H(\phi_{n+1}) dV - \int_{\Omega} \rho_a H(\phi_n) dV + \int_{t_n}^{t_{n+1}} \int_{\partial\Omega} \rho_a H(\phi) \boldsymbol{u} \cdot \mathbf{n} dS dt = 0 \tag{5}$$

Unfortunately, discrete solutions of eqn (1) do not necessarily satisfy eqn (5). This conservation error can be minor for highly resolved simulations over moderate time scales. However, coarse grid and/or long time simulations can lead to qualitatively incorrect results as the mass error accumulates to produce inaccurate fluid distributions, front speeds, and predicted loads.

A number of techniques have been developed to address this shortcoming in level-set methods. One approach is to improve the resolution of ϕ directly by solving eqn (1) to very high accuracy using Eulerian-Lagrangian approximations like the particle level-set method[5] or high-order discontinuous Galerkin schemes[10].

Other techniques introduce an explicit statement of mass conservation into the solution algorithm. For divergence-free velocity fields, eqn (1) can be rewritten as

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \boldsymbol{u}) = 0 \tag{6}$$

When approximated with a conservative discretization, this formulation conserves $\int_\Omega \phi dV$. If ϕ in fact approximates the Heaviside function, H, then the conserved quantity in eqn (6) approximates the fluid volume of one of the phases[12]. With this approach the approximate Heaviside function must be preserved in order to maintain the sharp interface approximation. Since non-oscillatory numerical methods for eqn (6) smear the profile over time, ϕ must be re-initialized by solving a nonlinear equation[12].

Similarly, the local volume fraction over a grid cell or mesh element can be defined as $V_e = \int_{\Omega_e} H(\phi) dV / |\Omega_e|$. A conservation law for the volume fractions then follows directly from eqn (5). While standard volume-of-fluid methods define V_e directly and conserve mass discretely, interface reconstruction remains a challenge, particularly for higher order and/or unstructured mesh approximations[6]. For this reason, techniques like the conservative level-set and volume-of-fluid (CLSVOF) approach combine a volume-of-fluid solution for V_e with solution of eqn (1) for ϕ to determine interface normals in the interface reconstruction process.

The hybrid method considered here also couples a level-set formulation with an (approximate) solution of eqn (5). The coupling is achieved via a correction, ϕ', defined at the time instant t^{n+1} as the solution of

$$\int_\Omega w \left[H(\phi^{n+1} + \phi') - \hat{H}^{n+1} \right] dV + \epsilon \int_\Omega \nabla w \cdot \nabla \phi' dV = 0 \quad \forall w \in W(\Omega) \tag{7}$$

where ϕ^{n+1} is the approximate solution of eqn (1) at time t^{n+1}, $\hat{H}^{n+1}$ is the approximate solution of eqn (5), and ϵ is a small parameter. Ideally, if the weighting space W is suitably chosen we have

$$\int_\Omega \left[H(\phi^{n+1} + \phi') - \hat{H}^{n+1} \right] dV = 0 \tag{8}$$

and ϕ' is in some sense a minimal, nearly constant perturbation of ϕ^{n+1}. The corrected approximation for ϕ then not only recovers mass conservation but also preserves geometric information like the signed distance property contained in ϕ^{n+1}.

While eqn (7) is nonlinear, the approach has several appealing features. It is suitable for finite element approximations of arbitrary order and topology as well as isogeometric or finite volume methods. Since it builds on basic level-set techniques, it can also be used to supplement existing level-set implementations to enforce mass conservation when necessary.

In the following, we formulate this approach for incompressible air/water flows and summarize a discrete approximation based on variational multiscale finite element methods[2]. We then present numerical results to illustrate performance for linear and quadratic approximations in two and three dimensions.

2 APPROACH

2.1 Continuous formulation

We begin with the standard level set formulation for incompressible two-fluid flow. Let Ω be the problem domain with boundary $\partial\Omega$, and ϕ be a level-set function that defines the two fluid domains. As discussed above, the zero level set of ϕ can be used to define the boundary between the two fluids through eqn (2) and its evolution is governed by eqn (1). To be specific, the water and air domains are defined as

$$\Omega_w = \{\boldsymbol{x} \mid \phi(\boldsymbol{x},t) > 0\}, \text{ and } \Omega_a = \{\boldsymbol{x} \mid \phi(\boldsymbol{x},t) < 0\}, \tag{9}$$

respectively. We let ρ_w and μ_w, and ρ_a and μ_a, denote the density and dynamic viscosity of water and air, respectively. Then, the density and viscosity of the two-fluid system are given by

$$\rho = \rho_w H(\phi) + \rho_a \left[1 - H(\phi)\right], \quad \text{and } \mu = \mu_w H(\phi) + \mu_a \left[1 - H(\phi)\right], \tag{10}$$

where $H(\phi)$ is given in eqn (4). In actual computations, H replaced by a regularized version, H_ε

$$H_\varepsilon(\phi) = \begin{cases} 0 & \text{if } \phi \le -\varepsilon; \\ \frac{1}{2}\left(1 + \frac{\phi}{\varepsilon} + \frac{1}{\pi}\sin\left(\frac{\phi\pi}{\varepsilon}\right)\right) & \text{if } |\phi| < \varepsilon; \\ 1 & \text{if } \phi \ge \varepsilon. \end{cases} \tag{11}$$

Conservation of fluid momentum and mass is expressed via a variable-coefficient Navier-Stokes system

$$\frac{\partial(\rho(\phi)\boldsymbol{u})}{\partial t} + \nabla\cdot(\rho(\phi)\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p - \nabla\cdot 2\mu(\phi)\nabla^s\boldsymbol{u} = \rho(\phi)\boldsymbol{f} \text{ in } \Omega \tag{12}$$

$$\nabla\cdot\boldsymbol{u} = 0 \text{ in } \Omega. \tag{13}$$

where p is the fluid pressure, $\boldsymbol{f}$ is the body force per unit mass, and ∇^s is the symmetric spatial gradient.

A minimal level-set formulation for two-phase incompressible flow consists of eqn (1), eqns (9)–(10), eqn (12), and eqn (13) along with appropriate initial and boundary conditions. Since they are based on a thin interface approximation, maintaining a sharp transition region in ϕ is critical for the accuracy of level-set methods. For this reason, level-set approaches typically include a "redistancing" step based on the solution of the Eikonal equation, which determines a signed distance function to the fluid-fluid interface[16,13]

$$\|\nabla \phi_d\| = 1, \ \phi_d = 0 \text{ on } \Gamma \tag{14}$$

As mentioned in the introduction, to this we add the differential form of the mass conservation equation for the volume fractions, eqn (5):

$$\frac{\partial \hat{H}}{\partial t} + \nabla \cdot \left(\hat{H} \boldsymbol{u} \right) = 0 \tag{15}$$

where density has been eliminated using incompressibility. Finally, the volume fraction is linked to the signed distance function with the nonlinear reaction-diffusion equation

$$\begin{aligned} \epsilon \Delta \phi' &= H_\varepsilon(\phi_d + \phi') - \hat{H} \\ \nabla \phi' \cdot \boldsymbol{n} &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{16}$$

where $\boldsymbol{n}$ is the unit outward normal to the two-fluid domain boundary, denoted by $\partial\Omega$, and ϵ is a parameter that penalizes the deviation of ϕ' from a global constant.

2.2 Numerical solution

To solve the system of equations that constitute the full mass-conservative approach, we use a first order splitting in time. To advance from time level t_n to t_{n+1} we proceed with the following steps[7]

1. Solve eqns (9)–(13) with $\rho(\phi^n)$ and $\mu(\phi^n)$ for $\boldsymbol{u}^{n+1}$,
2. Solve eqns (1) and (15) with $\boldsymbol{u}^{n+1}$ for ϕ_*^{n+1} and $\hat{H}_*^{n+1}$,
3. Solve eqn (14) with Γ defined by $\phi_*^{n+1} = 0$ for ϕ_d^{n+1}, and
4. Solve eqn (16) with ϕ_d^{n+1} and $\hat{H}_*^{n+1}$ for ϕ'.

We then set

$$\phi^{n+1} = \phi' + \phi_d^{n+1}, \text{ and } \hat{H}^{n+1} = H_\varepsilon\left(\phi' + \phi_d^{n+1}\right) \tag{17}$$

While this splitting decouples the full system, it still involves solution of several nonlinear PDE's that can be quite challenging in their own right. Our results here use residual-based variational multiscale methods[2]. Nonlinear shock-capturing is used to stabilize high Reynolds-number computations[3], while time integration is performed using either low-order variable coefficient backward difference formula (BDF) approximations or a generalized-α method.

3 Numerical Experiments

In the following, we consider two test problems to illustrate the behavior of our mass-conservation approach and compare its performance with a more standard level-set approach including redistancing. The first example is a well-known benchmark problem for interface propagation algorithms that tests both accuracy and conservation. The domain is the unit square, and the velocity is prescribed as[14]

$$u = \cos(\frac{\pi t}{8}) \sin(2\pi y) \sin^2(\pi x),\ v = -\cos(\frac{\pi t}{8}) \sin(2\pi x) \sin^2(\pi y) \tag{18}$$

The initial condition for the level set is a circle of radius 0.15 centered on $(0.5, 0.75)$, and the velocity field reverses in time to yield the initial conditions at the final time $T = 8$. Solutions were computed using continuous piecewise linear (P_1) and quadratic (P_2) finite element spaces on a series of refined grids starting with a base 21×21 node (L1) triangular mesh. Absolute mass conservation errors for a standard P_1 level set approximation with redistancing (NC) and the mass conserving (MC) solutions are shown in Figure 1 (left), while Figure 1 (right) shows the corresponding results for the P_2 approximation. Note that the P_1 and P_2 solutions were obtained using first and second order BDF approximations in time.

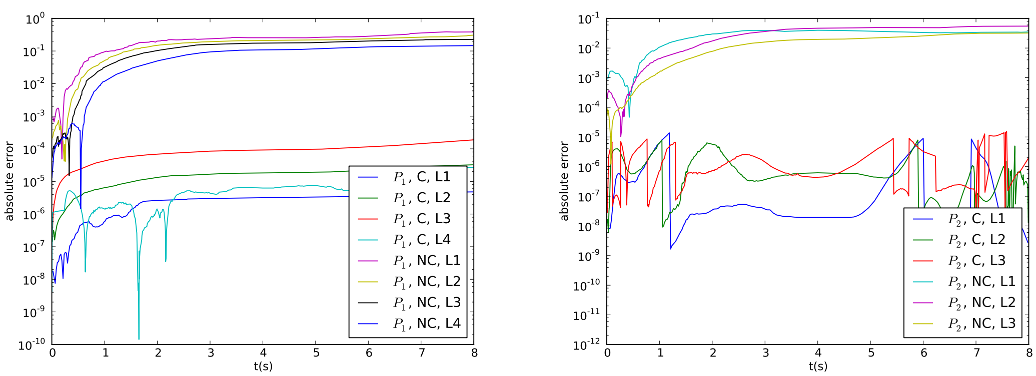

Figure 1: Vortex problem, absolute mass conservation errors for NC and MC methods. P_1 (left) P_2 (right)

For the second problem, the domain is a tank with dimensions 10m x 0.4m x 2m with a water column banked initially in the right side of the tank in a region 5m x 0.4m x 1 m. The time interval is $[0, 20s]$. The density of water is set to $\rho_w = 998.2kg/m^3$ and the dynamic viscosity to $\mu_w = 0.001kg/(ms)$. The density of air is set to $\rho_a = 1.205kg/m^3$ and the dynamic viscosity to $\mu_a = 0.0001kg/(ms)$. The fluid boundary conditions are no flow with free slip everywhere except the top of the tank, where a constant atmospheric pressure of zero is applied with transmission (outflow) boundary conditions. The time history of the absolute mass conservation error is given in Figure 2 (left), while a snapshot of the solution at $t = 2.0$ (s) is given in Figure 2 (right).

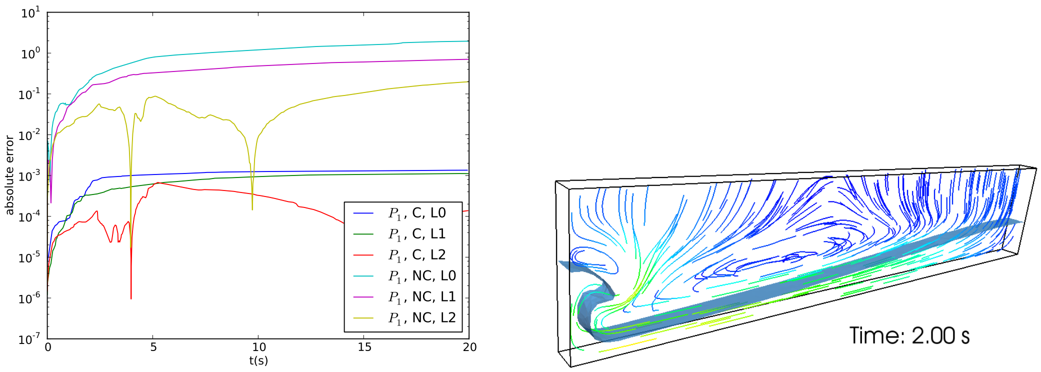

Figure 2: Problem 2, P_1 absolute mass conservation errors (left), free surface and velocity, $t = 2.0$ (right)

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