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# GROWTH OF MIXING LENGTH FOR THE TRACER FLOW PROBLEM IN A LONG-CORRELATED PERMEABILITY FIELD USING THE KARHUNEN-LOÈVE DECOMPOSITION APPROACH

**Marcio R. Borges***

*Laboratório Nacional de Computação Científica
Av. Getúlio Vargas,333, 25651-075 Petrópolis-RJ, Brazil
e-mail: mrborges@lncc.br, web page: http://www.lncc.br/

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**Summary.** The spatial variations in porous media, such as aquifers and petroleum reservoirs, occur at all length scales (from the pore scale to the reservoir scale) and are incorporated in the governing equations on the basis of random fields (geostatistical models). As a consequence, the velocity field is a random function of space. The randomness of the velocity field gives rise to a mixing region between fluids, which can be characterized by a mixing length $\ell = \ell(t)$. Under very general hypotheses, the scaling behavior of the mixing region is related to the scaling properties of the geological heterogeneities through relation $\ell(t) \sim t^{\gamma}$, where $\gamma = \max\{1/2,\ 1-\beta/2\}$ and $\beta$ is the Hurst coefficient. It gives a measure of relative importance of short vs. long length scale heterogeneities. In this work we use the theoretical result before to evaluate the representation of random fields using the Karhunen-Loève decomposition approach. For this we perform a numerical study for a statistical description of the mixing length, verifying the occurrence these scaling laws.The results shown that is necessary a large number of terms of the Karhunen-Loève to achieve a satisfactory approximation of the mixing length growth of the tracer flow in a fractally heterogeneous porous medium.

## 1 INTRODUCTION

The spatial heterogeneity of hydraulic properties of the natural porous media (aquifers and petroleum reservoir) produce important effects in the subsurface flows. These heterogeneities occurs at all length scales and, typically, its geological knowledge is much less detailed than is necessary to predict flow properties deterministically. Thus to model the uncertainty associated to flows we must use a stochastic approach. In this manner, several stochastic methods have been developed to deal with scarcity of field data.

The most conceptually straightforward method to solve stochastic equations is the Monte Carlo algorithm ($MC$) in which the relevant statistical moments are determined by averaging an ensemble of equally probable deterministic solutions[9;3]. Despite the Monte Carlo method is appealing due to its simplicity and has been widely applied in the numerical solution of a broad range of linear and nonlinear stochastic partial differential equations, it suffers from the drawbacks associated with the high computational costs involved with the large number of realizations often required to achieve statistical convergence and the fine grid resolution scales.

To overcome the high computational cost of Monte Carlo other methodologies have been proposed to solve equations governing flow in random media. In this sense, the Karhunen-Loève expansion ($KL$) has been widely used associated with finite element methods (Galerkin's numerical method) in order to reduce the complexity of the model by truncating the expansion at a suitable value[16]. Among them, we have the Spectral Galerkin method proposed by[7] and the Stochastic Collocation ($SC$) method[2]. However, the spectral Galerkin method suffers from the same large computational costs, since it generates non-sparse matrices associated with fully-coupled discrete equations and requires a large stochastic dimension in strongly heterogeneous media, so this numerical method may be as computationally expensive as $MC$[7;1;5].

The $SC$ method seeks an approximated solution in the truncated polynomial chaos space spanned by tensor product of orthogonal polynomials, from which a set of Gaussian collocation points is determined. This method is based in the affirmation that the source of randomness can be approximated using just a small number of uncorrelated, sometimes independent and these variables can be approximated in terms of base functions that depend of the zeros of orthogonal polynomials. In particular, whenever the random fields are expanded in a truncated Karhunen-Loève expansion, a particular choice of the basis for the tensor product space leads to the solution of uncoupled deterministic problems as in a Monte Carlo simulation[2;11;13]. The main advantage of solving uncoupled systems arising from $SC$ is the natural format for parallel computing. On the other hand, the number of uncoupled systems exponentially grows with the stochastic dimension (fact commonly referred as the *curse of dimensionality*).

In this paper our main objective is verify the use of $KL$ expansion, with reduced the complexity (truncated), to represent random fields with fractal covariance decaying in $MC$ studies of the tracer flow.

## 2 THE STOCHASTIC FLOW MODEL

We begin by presenting the mathematical model and the stochastic geology considered in this work.

### 2.1 Governing equations

The linear transport in porous media is governed by equation

$$\frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0, \tag{1}$$

where $s$ is the saturation of the tagged fluid and the random velocity field $\vec{v}$ is determined by Darcy's law and the incompressibility condition:

$$\vec{v} = -\frac{K}{\mu}\nabla p \quad \text{and} \quad \nabla \cdot \vec{v} = 0. \tag{2}$$

Here $K$ is a log-normal permeability field, whose properties are discussed below, $\mu$ is the fluid viscosity, and $p$ is the pressure.

In formulating (1), (2) we neglected the effects of gravity, compressibility and pore scale dispersion and set the porosity equal to a constant (it has been removed from the transport equation by rescaling the time variable).

For the numerical simulations we consider Eqs. (1) and (2) in a rectangular domain $\Omega = [0, L_x] \times [0, L_y]$, with boundary conditions

$$\begin{array}{rclcrcl} \vec{v}\cdot\vec{n} & = & -q, & \text{in} & x & = & 0, \\ p & = & 0, & \text{in} & x & = & L_x, \\ \vec{v}\cdot\vec{n} & = & 0, & \text{in} & y & = & 0 \text{ and } L_y, \end{array} \tag{3}$$

where $\vec{n}$ is the outward-pointing normal vector to $\partial\Omega$. The initial condition is given by

$$s(\vec{x}, t_0) = \begin{cases} 1 & \text{when} \quad x < 0, \\ 0 & \text{when} \quad x > 0. \end{cases} \tag{4}$$

The boundary conditions (3) simulate a flow predominantly parallel to the $x$ axis (left-to-right). The tagged fluid is injected uniformly (at a constant rate $q$) through the left vertical boundary ($x = 0$) of $\Omega$, which initially is saturated with untagged fluid. No flow conditions are imposed along the horizontal boundaries $y = 0$ and $y = L_y$. Fluid is produced from a well kept at constant (zero) pressure at the right vertical boundary ($x = L_x$).

### 2.2 Self-similar geology

We consider scalar, log-normal permeability fields

$$K(\vec{x}) = K_0 e^{\alpha Y(\vec{x})}, \quad K_0 > 0 \text{ fixed}, \tag{5}$$

where $\alpha$ is the heterogeneity strength that sets the overall strength of the fluctuations of the random field $K$. $Y(\vec{x})$ is a Gaussian random field characterized by its mean and covariance function. We consider a self-similar (or fractal)[9;10] model that introduce variability over all length scales. Thus the mean is an absolute constant, which we take to be zero, and the covariance is given by a power law:

$$\mathcal{C}_Y(\vec{x}, \vec{y}) = \langle \tilde{Y}(\vec{x})\tilde{Y}(\vec{y})\rangle = |\vec{x} - \vec{y}|^{-\beta}, \qquad \beta > 0. \tag{6}$$

Here angle brackets ($\langle\cdot\rangle$) denote ensemble averaging and $\tilde{Y} := Y - \langle Y\rangle$ is the stochastic deviation from the mean (fluctuation). Note that the fractal covariance (6) is singular at short distances and a cutoff $r_c > 0$ is necessary to regularize the fractal random field for sufficiently small lag-distances $r$. Thus, we can assume that the random fields $Y$ are stationary and isotropic. This implies that the mean is independent of $\vec{x}$ and the covariance depends only on the distance between two points, i.e. $\mathcal{C}_Y = \mathcal{C}_Y(r)$, where $r = |\vec{x} - \vec{y}|$ is the lag-distance.

The scaling exponent $\beta$ controls the degree of multiscale heterogeneity: As it decreases, the heterogeneities at the larger length scales are emphasized and the field becomes more regular (locally).

### 2.3 Karhunen-Loève Expansion (*KL*)

A random field (random process) can be represented as a series expansion involving a complete set of deterministic functions with corresponding random coefficients. The central idea is to describe a given statistical ensemble with the minimum number of modes.

The Karhunen-Loève procedure was proposed independently by[12] and[14] and is based on the eigen-decomposition of the covariance function. Then consider a random field (or stochastic process) $Y(\vec{x}, \omega)$ defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ composed of the sample space, the ensemble of events and a probability measure, respectively, and indexed on a bounded domain $\Omega$. The process $Y$ can be expressed as

$$Y(\vec{x}, \omega) = \langle Y(\vec{x}) \rangle + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \phi_i(\vec{x}) \xi_i(\omega), \tag{7}$$

where $\lambda_i$ and $\phi_i$ are the eigenvalues an eigenfunctions of the covariance function $\mathcal{C}_Y(\vec{x}, \vec{y})$. By definition, $\mathcal{C}_Y(\vec{x}, \vec{y})$ is bounded, symmetric and positive definite and has the following eigen-decomposition:

$$\mathcal{C}_Y(\vec{x}, \vec{y}) = \sum_{i=1}^{\infty} \lambda_i \phi_i(\vec{x}) \phi_i(\vec{y}). \tag{8}$$

The eigenvalues and eigenfunctions of Eq. (8) are the solution of the homogeneous Fredholm integral equation of second kind given by

$$\int_{\Omega} \mathcal{C}_Y(\vec{x}, \vec{y}) \phi_i(\vec{x}) d\vec{x} = \lambda_i \phi_i(\vec{y}). \tag{9}$$

## 3 MIXING LENGTH GROWTH

The growth of macroscopic mixing region, induced by the random velocity field, of the tracer fronts flowing in a fractally heterogeneous porous medium is shown to be anomalous in cases dominated by slowly decaying correlations[8].

For the tracer flow problem (1)-(4), (5)-(6), a characterization of the long-time asymptotic growth of the mixing region was obtained in[8;17] within distinct analytic approximations at the level of perturbation theory. Theory of fluid mixing provides the following scaling law for the mixing length $\ell = \ell(t)$, which characterizes the size of the region where the fluids mix macroscopically:

$$\ell(t) = \mathcal{O}(t^\gamma), \quad \text{with} \quad \gamma = \max\left\{\frac{1}{2}, \frac{1}{2} + \frac{1-\beta}{2}\right\}. \tag{10}$$

Two qualitatively distinct regimes occur: if $\beta > 1$, then $\gamma = 1/2$, and the mixing process is Fickian; if $1 > \beta > 0$, then $\gamma > 1/2$, and the mixing is anomalous (i.e., the diffusivity increases with time or, equivalently, with travel distance).

This scaling law is valid in the limit of small velocity fluctuations, and provides quantitative agreement with numerical simulations for a range of moderately large permeability heterogeneities[9;3].

## 4 NUMERICAL EXPERIMENTS

In this section we conduct a numerical experiment to illustrate the effect of truncation of the *KL* expansion on the mixing length growth. For this purpose, we perform Monte Carlo simulations conducted over ensembles of 1 000 realizations of the permeabilities (generated with different $M$ values) to compute the mixing length in accordance with[6;3]. Hence, for each ensemble of realizations of the permeability field $K$, the Eqs. (1) and (2), subject to boundary (3) and initial condition (4), were solved numerically by an accurate simulator which combines the mixed finite element method[4], for the solution of the velocity-pressure equation, with a second order, non-oscillatory central finite difference scheme[15], for the solution of the saturation equation. The input data used in the experiments were: $L_x = L_y = 1.0$, Hurst coefficient $\beta = 0.5$, $q = 1.0$ and $\alpha = 0.5$.

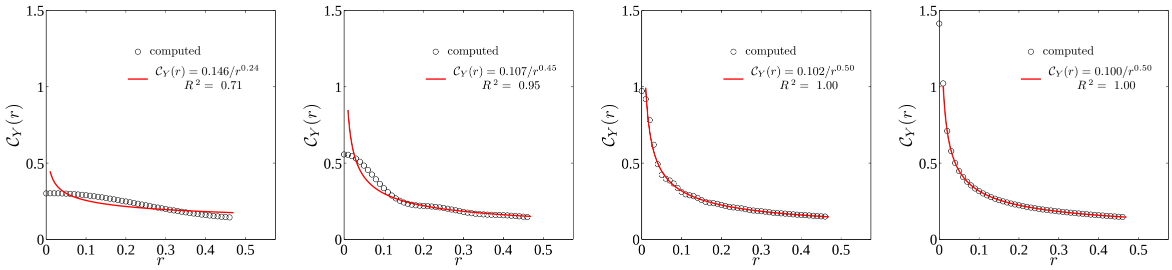
The computed saturation profiles $s = s(\vec{x}, t)$ are then averaged in the direction transverse to the mean flow (the $y$-direction) and across the ensemble to define $\bar{s} = \bar{s}(x, t)$, which is used to compute numerically the mixing length (see[3]):

$$\ell(t) = 2\sqrt{\pi} \int_{L(t)}^{L_x} \bar{s}(x, t)\, dx. \tag{11}$$

Here, $L_x$ is the length of computational domain in the horizontal direction $x$ and $L(t) = \langle \vec{v} \rangle t$ is the mean distance traveled by the heterogeneous front over the time period $t$.

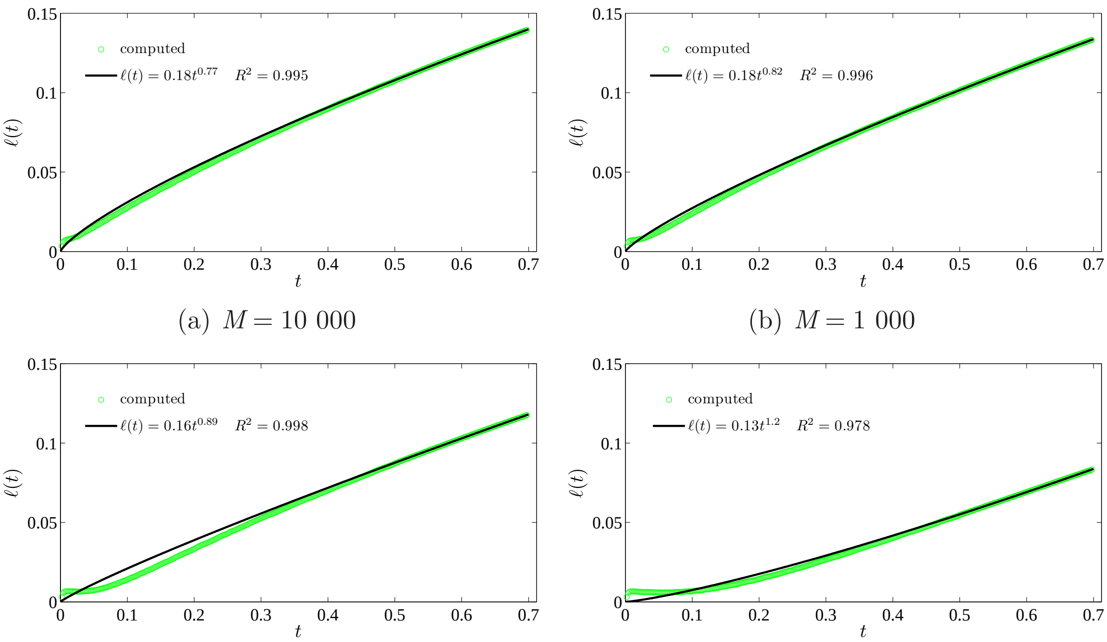
Realizations of the Gaussian random field $Y$ ($\beta = 0.5$ and $100 \times 100$ mesh) were generated numerically using the *KL* expansion (7) truncated at $M = 10$, 100, 1 000 and 10 000 terms. For this the Fredholm integral (9) was solved with a Galerkin method.

We highlight that the randomness must be realistically represented to achieve a meaningful solution to the mixing length growth problem. Hence, in order to check the quality of the random fields we computed the covariance of the ensemble (1 000 realizations). In Figure 1 we display the computed covariance (in $x$-direction) as a function of the lag-distance $r$. Solid curves are adjusted by a least squares fit and the respective regression coefficients $R^2$ are shown. The results exhibit a good agreement between the theoretical behavior ($\beta = 0.5$) and the numerical one only for $M = 1\ 000$ and $M = 10\ 000$. Small values of $M$ were not sufficient to represent the covariance function in a satisfactory way. This poor representation of the geology obtained with small $M$ values will affect the mixing length growth.



(a) $M = 10$ (b) $M = 100$ (c) $M = 1\ 000$ (d) $M = 10\ 000$

Figure 1: Power-law covariance as a function of the lag-distance $r$ in the $x$-direction using a $100 \times 100$ geological mesh ($\beta = 0.5$).



(a) $M = 10\ 000$ (b) $M = 1\ 000$

(c) $M = 100$ (d) $M = 10$

Figure 2: The mixing length as a function time ($\beta = 0.5$).

Figure 2 shows the mixing length in terms of time. The solid curves are adjusted by a least squares fit. The initial points (small times) were not considered because the mixing function is dominated by numerical error. The scaling exponent ($\gamma = 0.77$ in Figure 2(a)) obtained for the expansion truncated with $M = 10\ 000$ exhibit a good agreement with the theoretical prediction ($\gamma = 0.75$ in Eq. (10)). Here we emphasize that the theoretical prediction are restricted to the regime of small fluctuations and to asymptotic times. The random fields generated with $KL$ expansion truncated at $M$ values 10, 100 and 1 000 were not effective to provide a satisfactory representations of the stochastic model of the tracer flow.

A visual comparison between the mixing lengths obtained with different $M$ values is given by Figure 3. Mixing length growth was underestimated by the experiments with small $M$ values. Note that when $M$ increases the mixing length converges to the one obtained with $M = 10\ 000$. The discrepancy of the results suggest that a good approximation of stochastic problem with fractal geologies only will be obtained with a large number of terms of the $KL$ expansion which makes computationally infeasible methods such as Stochastic Collocation and Spectral Galerkin due the "curse of dimensionality".

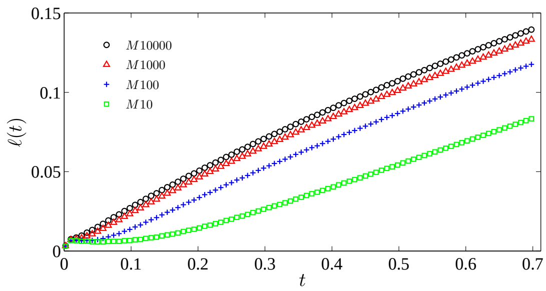



Figure 3: The mixing length as a function time ($\beta = 0.5$).

## 5 CONCLUSIONS

- The results shown that is necessary a large number of terms of the Karhunen-Loève to achieve a satisfactory approximation of the mixing length growth of the tracer flow in a fractally heterogeneous porous medium.

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