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UPSCALING RESERVOIR SIMULATION USING MULTILEVEL OPERATOR COARSENING

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1 INTRODUCTION

Thanks to new seismic and geostatistical techniques, reservoir geological models are continuously improving at defining details and including complex features inside the porous medium. This is happening at such a fast pace that for all the advancement in CPU computational powers, reservoir simulators cannot properly deal with them, leading to an immediate cause for developing methods to compress data into efficient averaged representations. Averaging techniques must give similar production forecasts as those from detailed geological models. The procedure to give effective data representing the detailed data or more generally any method that releases the burden of using computationally expensive algorithms is called upscaling. A good upscaling method zooms out a number of scales in resolution of data and yet gives similar production forecasts to the original fine grid model.

In this work we try to assess the applicability of a wavelet approach in the calculation of effective permeability following on from previous works[1,2] by compressing the fine operators. Elsewhere[3,4] wavelet upscaling approaches have been used for upscaling by adaptive grid construction, however the framework of upscaling through fine operator compressing is fairly recent and needs more attention.

Wavelets are although a relatively young field of mathematics. They have a rich mathematical connection with transformation analysis on the one hand, and multigrid numerical methods on the other. The idea of conveying an equation of residuals from a fine grid to coarse grid (two-grid method) to dampen the oscillations is a very well sought-after approach in solution of large scale system of equations.

In a similar fashion to multigrid and under the renormalisation class of upscaling methods, we have investigated the applicability of wavelets to compress equations. For that purpose, the pressure equation is discretized, compressed and is used to avoid the computationally expensive step of inversion in reservoir simulation. This applicability is examined through some numerical experiments. Pros and cons and the possibility to extend the method over real flow problems are discussed.

2 WAVELETS

The Fourier Transform (FT) uses the sine and cosine waves as the basis of transformation of an original function f that span throughout the entire time or space domain. Once we integrate the function f the features which are localised in real space are evenly spread across the whole frequency. This is a main drawback of FT, henceforth we cannot expect transformation of a function with short-lived suddenly-changing behaviour without the effect on other frequencies. Consequently all Fourier coefficients may suffer from the local effects. Therefore localised basis functions are better representation of data with local changes. A very well-established localised transform is Wavelet Transform[5].

The wavelet transform besides the concept of the "translation" or shift of basis in physical domain, "scale" is also considered. First, a function can be transformed by basis functions and their shifts in domain throughout space. Then scale is changed. We can either lower or improve the resolution. Based on basis functions of the new scale and their shifts, transformation of the function is carried out again. It turns out that this sort of scale analysis is less sensitive to noise by approximating the average of fluctuations (high frequencies) at different scales; [see e.g. Ref. [6]]. These new transforms are called wavelets since they behave like small waves throughout the domain and henceforth they are localised.

2.1 Haar wavelet

The simplest scaling and wavelet function used in this work is the so-called box or Haar function[7]:

$$\varphi(x) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{elsewhere} \end{cases}, \qquad \psi(x) = \begin{cases} 1 & 0 \le x < 1/2 \\ -1 & 1/2 \le x < 1. \\ 0 & \text{elsewhere} \end{cases} \tag{1}$$

Considering a function $f(x)$ defined on interval of $[0,1]$, we can write:

$$f(x) = \left(\int f(x)\, \varphi(x) dx \right) \varphi(x) + \left(\int f(x)\, \psi(x) dx \right) \psi(x). \tag{2}$$

Generally for a function on an infinite domain we can translate or shift scaling and wavelet functions and tile the domain and carry out integration over all tiles. The first integral takes the averages over the interval. The second term takes the differences over the same interval. Here, the coefficients for the averages and differences are $\langle f, \varphi \rangle = \int f(x)\, \varphi(x) dx$ and $\langle f, \psi \rangle = \int f(x)\, \psi(x) dx$, respectively.

In discrete form, continuous functions $\varphi(x)$ and $\psi(x)$ are written as $\varphi = [1\ 1\ 0\ 0 \dots]$, and $\psi = [1\ -1\ 0\ 0 \dots]$. Then by shifting nonzero entries of these vectors successively throughout the domain, we can cover all the entire entries of an original dataset. Thus, instead of integration, dot products will evaluate averages and differences. For example, supposing a pressure vector $\boldsymbol{P} = [p_1, p_2]$ for two gridcells, then for a and d as coefficient of averages and differences by $\varphi = [1\ 1], \psi = [1\ -1]$ we have:

$$a = \boldsymbol{P}.\boldsymbol{\varphi}/2 = (p_1 + p_2)/2\,, \quad d = \boldsymbol{P}.\boldsymbol{\psi}/2 = (p_1 - p_2)/2. \tag{3}$$

To recover $\boldsymbol{P}$, we have $\boldsymbol{P} = a.\boldsymbol{\varphi} + d.\boldsymbol{\psi}$. Notice that $1/2$ is added for normalisation. We can also write:

$$\boldsymbol{w} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, \tag{4}$$

$\boldsymbol{w}$ is a transformation matrix comprising of first row as averaging and second row as differencing operators. The inverse transform $\boldsymbol{P} = \boldsymbol{w^{-1}P'}$ recovers the original function.

For vectors of sizes greater than two, first the averages over each pair are calculated, and then we can transform new averages into a new set of average and difference coefficients. In this fashion we will have the hierarchical expansion of a vector $\boldsymbol{P}(.)$ in fine grid space $\boldsymbol{J}$ to coarser grid $\boldsymbol{j}$ as:

$$\boldsymbol{P}(.) = \sum_{k\in\mathbb{Z}} a_{j,k}\boldsymbol{\varphi}_{j,k}(.) + \sum_{m=j}^{J-1}\sum_{k\in\mathbb{Z}} d_{m,k}\boldsymbol{\psi}_{m,k}(.). \tag{5}$$

In Eq.(5), (.) denotes the domain, that can be a vector of N gridcells. k is the number of pairs, for example for the first transformation $k=N/2$. j and J are respectively coarse and fine grids. a and d are respectively average and difference coefficients. It is clear that we have $\boldsymbol{P}(.)$ as a series of a coarse set of average coefficients and a hierarchy of difference coefficients from coarse to fine grids.

Extension of wavelets to higher dimensions is generally carried out by using the tensor product of one dimensional basis functions to cover dimensional domain of an original data. In two dimensions we produce three groups of wavelets corresponding to scaling function $\boldsymbol{\Phi_{j,k,k'}}$. One represents differences in the horizontal direction, one in the vertical and the other in the diagonal direction. If h stands for horizontal, v for vertical and d for diagonal differences, the wavelets are $\boldsymbol{\psi}^{\mathbf{h}}_{j,k,k'}$, $\boldsymbol{\psi}^{\mathbf{v}}_{j,k,k'}$, and $\boldsymbol{\psi}^{\mathbf{d}}_{j,k,k'}$.

2.2 Matrix representation of transforms

Writing the filters as entries of a vector in terms of discrete numbers and then shifting the vectors through a matrix rows will give us a form of mapping operator. This operator is applicable to vectors of similar size and maps that vector into two sets of scaling and differences (or detail) coefficients. In general we can write: $\boldsymbol{w}_j = \begin{bmatrix}\Phi_j\\ \Psi_j\end{bmatrix}$, which Φ_j and $\Psi_j$represent discrete values of $\varphi(x)$ and $\psi(x)$ as averaging and differencing operators in scale j respectively. From orthonormality of basis functions and consequently each rows of $\boldsymbol{w}_j$, it is simple to investigate that $\boldsymbol{w}_j^*\boldsymbol{w}_j = \boldsymbol{I}$, where $\boldsymbol{I}$ is the identity matrix. One can now go further into coarser scales and come up with a hierarchy of high and low pass filters corresponding to a ladder of scales inside a single matrix, to do so we write:

$$\boldsymbol{m}^{\ell} = \begin{bmatrix}\boldsymbol{w}^{j-\ell} & \\ & \boldsymbol{I}\end{bmatrix}\begin{bmatrix}\boldsymbol{w}^{j-\ell+1} & \\ & \boldsymbol{I}\end{bmatrix}\dots\begin{bmatrix}\boldsymbol{w}^{j-1} & \\ & \boldsymbol{I}\end{bmatrix}\boldsymbol{w}^{j}. \tag{6}$$

In this equation ℓ is the coarsest level and j is the finest one, the size of the corresponding $\boldsymbol{I}$ blocks are so that matrix multiplication can be carried out. Note that hierarchical matrices or

$\boldsymbol{m}$-matrices are also unitary i.e. their transpose are their inverse. The resolution in which a data set satisfactorily and efficiently expresses the original data in compressed form can be chosen by thresholds. The hierarchical transformation in this fashion can be extended over discretized form of operators from partial differential equations. In this sense we actually project an operator into coarse and detail subspaces. Consider the following one-dimensional boundary value problem partial differential equations:

$$\boldsymbol{\mathcal{L}}u(x) = f \ \ on\, \Omega, \ f = \mathrm{g}\ on\, \partial\Omega. \tag{7}$$

$\boldsymbol{\mathcal{L}}$ is differential operator. After discretization of the equation in fine grid scale $j+1$ we have:

$$\boldsymbol{L}_{j+1}\boldsymbol{U} = \boldsymbol{F}, \quad \boldsymbol{U}, \boldsymbol{F} \in j+1, \boldsymbol{L}_{j+1} \in \boldsymbol{\mathcal{L}}(j+1, j+1). \tag{8}$$

This equation can originate from any discretization scheme of a given differential equation. Now, using orthogonality of $\boldsymbol{w}$ matrices we can write: $(\boldsymbol{w}_j\boldsymbol{L}_{j+1}\boldsymbol{w}_j^*)\boldsymbol{w}_j\boldsymbol{U} = \boldsymbol{w}_j\boldsymbol{F}$, Which the projected operator and vectors are:

$$\boldsymbol{w}_j\boldsymbol{L}_{j+1}\boldsymbol{w}_j^* = \begin{bmatrix} H_j\boldsymbol{L}_{j+1}H_j^* & G_j\boldsymbol{L}_{j+1}H_j^* \\ H_j\boldsymbol{L}_{j+1}G_j^* & G_j\boldsymbol{L}_{j+1}G_j^* \end{bmatrix} = \begin{bmatrix} \boldsymbol{L}_j & \boldsymbol{B}_j \\ \boldsymbol{C}_j & \boldsymbol{D}_j \end{bmatrix}, \boldsymbol{w}_j\boldsymbol{U} = \begin{bmatrix} U_a \\ U_d \end{bmatrix}, \boldsymbol{w}_j\boldsymbol{F} = \begin{bmatrix} F_a \\ F_d \end{bmatrix}. \tag{9}$$

Which *a* stands for average and *d* stands for differences or details. The application of another level of transform is equivalent to using a two-level hierarchical matrix $\boldsymbol{m}$ on the original equation as:

$$\left(\boldsymbol{m}\boldsymbol{L}_{j+1}\boldsymbol{m}^*\right)\boldsymbol{m}\boldsymbol{U} = \boldsymbol{m}\boldsymbol{F}. \tag{10}$$

One can perform multilevel hierarchical matrices to have multilevel representation of the solution in this manner. By calculating a coarse solution we can consider next level higher-resolution operator to calculate finer solution. The advantage is the ability to decide significance of details in each level so that we can stop the algorithm at some certain stage.

3 UPSCALING OF THE PRESSURE EQUATION

In this section we carry out the above matrix procedure on the discretized pressure equation, the resulting coarse submatrix is used for calculation of coarse fluxes and coarse saturation like any other absolute permeability upscaling process. The two dimensional 5-point-stencil-finite-difference discretization of the elliptic pressure equation for flow of an incompressible fluid in a no-gravity, no-capillary porous media reads:

$$-\nabla.\lambda(s)\mathbf{K}\nabla p = q \ \rightarrow \ \boldsymbol{T}p = f. \tag{11}$$

In this equation $\lambda(s)$ is the total mobility factor as a function of saturation, $\boldsymbol{K}$ is diagonal permeability tensor, p is the total pressure and q includes source and sink and boundary conditions contributions. After finite difference discretization $\boldsymbol{T}$ is the diagonal matrix containing transmissibilities over the grid block faces. The transmissibilities in finite difference are constructed by harmonic averaging of $\lambda(s)\mathbf{K}$ between adjacent gridcells. The

same multiscale representation described in previous sections can be followed by application of hierarchical matrix $\boldsymbol{m}$ as:

$$(\boldsymbol{mTm}^{*})\boldsymbol{mp} = \boldsymbol{mf}. \tag{12}$$

Defining the multiscale operator as $\boldsymbol{T}^{ms} = \boldsymbol{mTm}^{*}$, and multiscale vectors $p^{ms} = \boldsymbol{m}p$, $f^{ms} = \boldsymbol{m}f$ leading to multiscale representation of pressure equation as:

$$\boldsymbol{T}^{ms}p^{ms} = f^{ms} \tag{13}$$

Now, selecting the coarse part of the operator (see **Figure 1**) is equivalent to upscaling the whole equation. After coarsening the operator, the relevant coarse diagonal transmissibilities are extracted to calculate the flux.

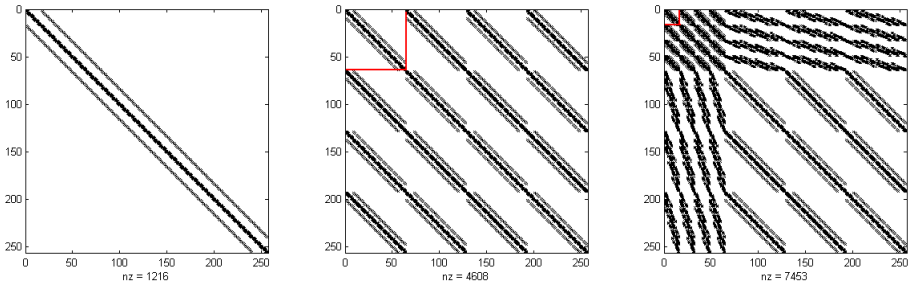

Figure 1-Non-zero sparsity pattern of an original fine pentadiagonal transmissibility matrix (left) and its decomposition into two levels of projection (middle and right). In each projection, the block upper left block separated by red lines represent the compressed form of the operator, a self-similar repeating sparsity pattern is visible; such pattern has been taken advantage of in large scale code developing

4 NUMERICAL EXPERIMENT: INCOMPRESSIBLE TWO-PHASE TWO-DIMENSIONAL WATER FLOODING

In this section we apply the method for upscaling to a water flooding process. A permeability field has been generated using a correlated log-normal permeability[8]. We run simulations on a generated reservoir cross-section of 256×256 number of gridcells with an anisotropic permeability ($k_x \neq k_y$). The subsequent permeability field has a contrast in absolute permeability of order of magnitude of 6, the mean and standard deviation of the original grid upon which the log-normal permeability was constructed are $\mu = 1$ and $\sigma^2 = 25$, respectively. The dimensionless correlation lengths are $l_x \approx 0.18, l_y \approx 0.08$ in unites of the linear dimensions of the system. The natural logarithm of permeability field is shown in **Figure 2**. The flow is taken to be horizontal. A a quarter 5-spot pattern, water is injected in upper left gridcell while oil is produced from lower right gridcell. For these two wells we used a constant rate of injection and production of one pore volume per unit of time. No flow boundary condition is used elsewhere in the domain. The simulation was run for the injection of one pore volume. **Table 1** lists other specifications.

Number of gridcells	256*256
Area	1
Dimensionless Correlation length in x-direction	0.18
Dimensionless Correlation length in y-direction	0.08
$\boldsymbol{\mu/\sigma^2}$	0.04
$\boldsymbol{k_{ri}}$	$s_i^2, i = o, w$
$\boldsymbol{\mu_o/\mu_w}$	1
φ	1

Table 1-Specifications for first example

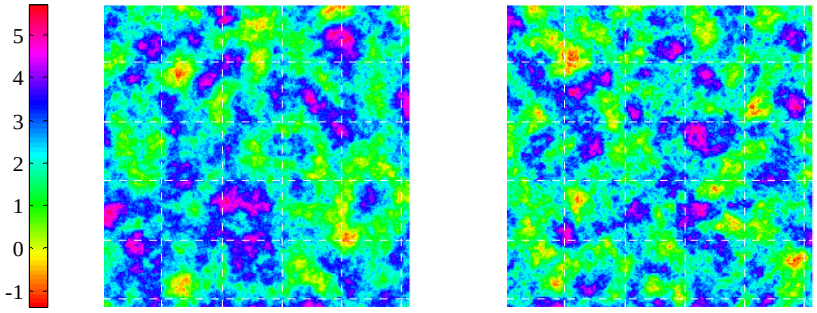

Figure 2- The absolute permeability, left is logarithm of permeability in **x** and second is in **y** direction, respectively.

After each coarse pressure and saturation solve, we carry out propagation of saturation over the original fine grid with *cubic spline interpolation*. This choice of interpolation provides a smoother profile. The operator-coarsening has been assessed in three levels and the saturation profiles as well as the production curves are compared to the reference fine model in **Figure 3**. The overall shape of the cuts and the breakthrough times with up to two levels are in good agreement with the fine model.

In **Table 2** scale-up factor refers to the value that operator is compressed with respect to the original fine operator. Computation time represents runtime in seconds achieved by a machine with 2.80 GHz CPU. $|\varepsilon_{RF}|$ denotes the relative error in recovery factor at one pore volume injected. $|\varepsilon_{BT}|$ denotes the relative error in breakthrough time. Breakthrough time is chosen as the time that oil production fraction falls to 0.95.

The second example is a comparative experiment between an operator coarsening and permeability averaging upscaling. For emphasizing the underlying permeability heterogeneities, a shale structure of zero permeability is placed diagonally across the flow path. **Figure 4** is the saturation profile for the fine model and the upscaled models, it is clear that the operator coarsening has led to a closer approximation of saturation to the fine model than that by just averaging the permeability.

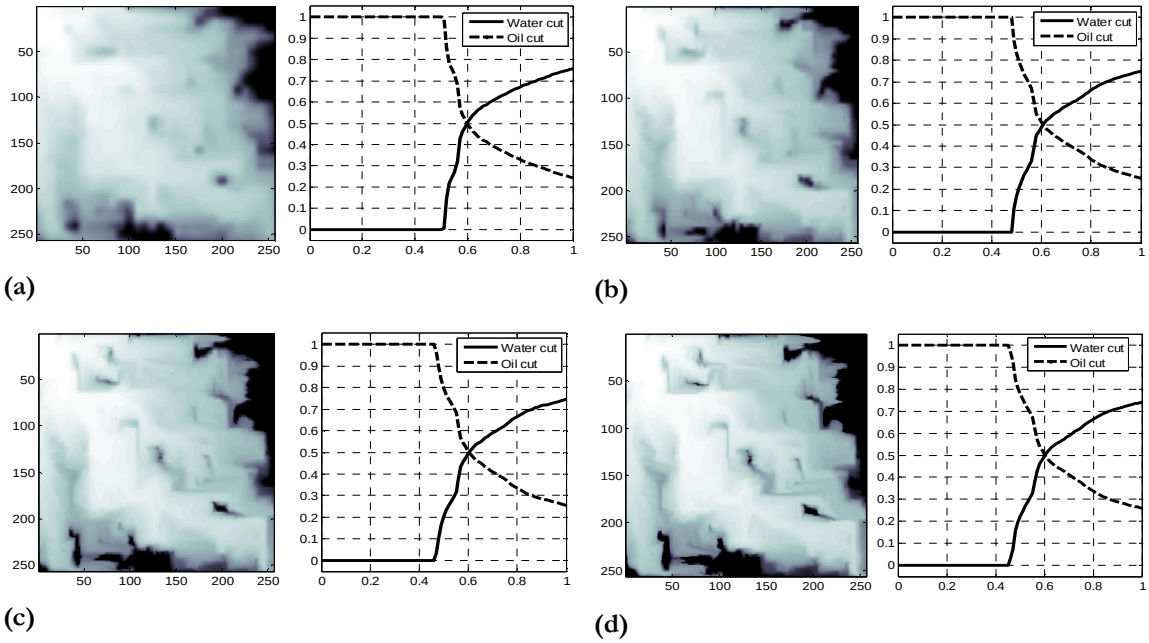

Figure 3- Simulation results for one pore volume injection in 3 levels of coarsened operator **(a,b,c)** and reference model **(d)**, a is Level Three, **(b)** is Level Two and **(c)** is Level One, in each subplot the saturation profile over the cross-section and curves for water and oil cuts are depicted, resolution is enhanced level by level

Level	Scale-up factor	Computation time(sec.)	$\|\varepsilon_{RF}\|$	$\|\varepsilon_{BT}\|$
REF.	---	826	---	---
1	4	74	1	2.2
2	16	33	1	6.6
3	64	30	3	13.3
4	256	29	8	22.2

Table 2-Quality assessment for four levels of operator-coarsening

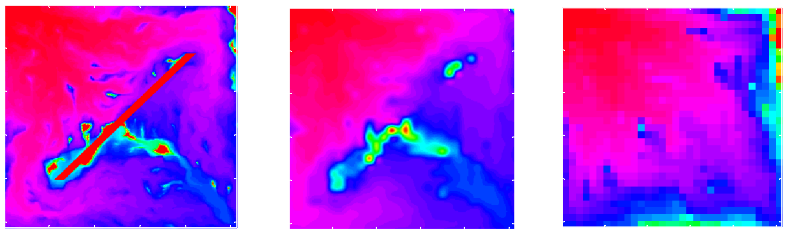

Figure 4- A shale-included system: fine scale saturation profile at 1 PV injection (left), same profile for third level of operator-coarsening (middle) and same profile for third level of permeability transformation by Haar scaling 2d functions (right)

5 CONCLUSIONS

In this work an extension to the upscaling procedure presented in [1] is given. A self-repeating pattern of wavelet matrices provides us the opportunity to extend the method for systems as large as 60 thousand gridcells. We emphasised the preservation of the fine scale permeability field. Instead of averaging the absolute permeabilities, the discretized operator was transformed into a compressed, average representation. Moreover, inspired by multigrid methods we implemented a high order prolongation operator which can map the saturation profile back to an original fine grid in more monotonic result than what a linear or bilinear interpolation could achieve.

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