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# 3-D MODELING OF COUPLED SUBSURFACE FLOW-STRESS BY MIXED FINITE ELEMENTS

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**Summary.** The accurate simulation of coupling between flow and stress in saturated porous media is a major issue in a broad variety of fields, ranging from reservoir engineering to biomechanics. Despite the intensive research carried out in recent years, the numerical solution to the partial differential equations governing the behaviour of real fluid saturated heterogeneous porous media still represents a demanding task. In the present communication an original fully coupled 3-D Mixed Finite Element (MFE) model is developed with the aim at reducing the numerical oscillations of the pore pressure predicted with the aid of traditional FEs. Using a mixed approach for the flow equation enforces an element-wise conservative velocity field with a similar order of approximation for both pore pressure and stress. This helps stabilize the numerical solution and obtain a more accurate calculation of the fluxes. The MFE model is validated against Terzaghi's analytical solution and successfully tested in two large size realistic and computationally challenging applications, i.e. the consolidation of a river embankment and the Noordbergum effect due to groundwater withdrawal from a shallow confined aquifer.

## 1 INTRODUCTION

Poro-elasticity denotes the coupled process between mechanics and flow in porous media. Its theoretical basis goes back to the mid 1920s when Terzaghi described analytically the one-dimensional (1-D) consolidation of a soil column under a constant load [1]. In 1941 Biot generalized Terzaghi's theory to three-dimensional (3-D) porous media [2] by establishing the mathematical framework which is usually termed as poro-elasticity. Despite the intensive research in the area, the numerical solution of the governing PDEs, however, can be a difficult task mainly because of two factors. First, the solution of the fully coupled problem tipically involves large algebraic systems that can be severely ill-conditioned [3]. Second, the pore pressure can be numerically unstable, the main reason being the assumption of a nearly incompressible fluid which may yield a locking phenomenon with traditional FEs [4]. Different remedies can be implemented to cope with such numerical difficulties. Recently some approaches have been advanced based on mixed formulations, allowing for both solving nearly incompressible problems with no locking and a greater flexibility in predicting independently pressures, displacements and fluxes, e.g. [5, 6].

In the present communication a fully coupled 3-D MFE formulation is developed to solve numerically the Biot equations of consolidation with the aim at alleviating the instabilities in the pore pressure calculation. The fluid pore pressure and flux are approximated in the lowest order Raviart-Thomas mixed space, while linear tetrahedral FEs are used for the displacements. The main reasons for the above choices are threefold. First, keeping the flux as a primary variable allows for a greater accuracy in the velocity field compared to Galerkin FEs, which can be of interest whenever a consolidation model is coupled with an advection-diffusion equation, e.g. to account for thermal effects or contaminant transport. Second, a mixed formulation for the flow problem is element-wise mass conservative because the normal flux is continuous across the element boundaries. Third, the practical advantages from using low-order interpolation elements, such as ease of implementation, refinement, and discretization of geometrically complex and heterogeneous domains, are thouroughly preserved. The MFE model is verified against the well-known Terzaghi's analytical solution, and tested in two large size realistic and numerically challenging applications.

## 2 MFE MODEL OF BIOT CONSOLIDATION

The interaction between a granular material and the fluid filling its pores is governed by a stress equilibrium equation coupled to a mass balance equation, with the relationship linking the grain forces to the fluid pore pressure based on Terzaghi's effective stress principle. The equilibrium equation for an isotropic poro-elastic medium incorporating the effective stress concept reads:

$$\mu \nabla^2 \hat{\mathbf{u}} + (\lambda + \mu) \nabla \mathrm{div} \hat{\mathbf{u}} = \alpha \nabla p + \mathbf{b} \tag{1}$$

where $\lambda$ and $\mu$ are the Lamé constants, $\alpha$ is the Biot coefficient, $\mathbf{b}$ the body forces, $\hat{\mathbf{u}}$ the medium displacements and $p$ the fluid pore pressure. The fluid mass balance is prescribed by the continuity equation:

$$\mathrm{div}\mathbf{v} + \frac{\partial}{\partial t} (\phi\beta p + \alpha \; \mathrm{div}\hat{\mathbf{u}}) = f \tag{2}$$

where $\phi$ is the medium porosity, $\beta$ the fluid compressibility, $t$ time, $f$ a flow source or sink and $\mathbf{v}$ the Darcy velocity. Equation (2) must be coupled with Darcy's law defining $\mathbf{v}$:

$$\overline{\boldsymbol{\kappa}}^{-1}\mathbf{v} + \nabla p = 0 \tag{3}$$

with $\overline{\boldsymbol{\kappa}} = \overline{\mathbf{k}}/(\rho g)$, $\overline{\mathbf{k}}$ the hydraulic conductivity tensor and $(\rho g)$ the fluid specific weight.

Equations (1) through (3) form a coupled partial differential system defined on a 3-D domain $\Omega$ with boundary $\Gamma$ and $\hat{\mathbf{u}}$, $\mathbf{v}$ and $p$ as unknowns. This system can be solved with appropriate boundary (BCs) and initial (ICs) conditions:

$$\text{BCs:}\ \begin{cases} \hat{\mathbf{u}}(\mathbf{x},t) = \hat{\mathbf{u}}_D(\mathbf{x},t) & \text{over } \Gamma_D \\ \overline{\boldsymbol{\sigma}}_{\text{tot}}(\mathbf{x},t)\,\mathbf{n}(\mathbf{x}) = \mathbf{t}_N(\mathbf{x},t) & \text{over } \Gamma_N \\ p(\mathbf{x},t) = p_D(\mathbf{x},t) & \text{over } \Gamma_p \\ \mathbf{v}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) = q_N(\mathbf{x},t) & \text{over } \Gamma_q \end{cases} \qquad \text{ICs:}\ \begin{cases} \hat{\mathbf{u}}(\mathbf{x},0) = \hat{\mathbf{u}}_0(\mathbf{x}) \\ p(\mathbf{x},0) = p_0(\mathbf{x}) \end{cases} \tag{4}$$

with $\Gamma_D \cup \Gamma_N = \Gamma_p \cup \Gamma_q = \Gamma$, $\overline{\boldsymbol{\sigma}}_{\text{tot}}$ the total stress tensor, $\mathbf{n}$ the outer normal to $\Gamma$ and $\mathbf{x}$ the position vector in $\mathbb{R}^3$, while the right-hand sides are known functions.

Approximate the medium displacement in space with continuous piecewise linear polynomials $\ell_i$, $i = 1, \ldots, n_n$, with $n_n$ the number of FE nodes in $\Omega$:

$$\hat{\mathbf{u}}(\mathbf{x},t) \simeq \left[ \sum_{i=1}^{n_n} \ell_i(\mathbf{x})\, u_{x,i}(t),\ \sum_{i=1}^{n_n} \ell_i(\mathbf{x})\, u_{y,i}(t),\ \sum_{i=1}^{n_n} \ell_i(\mathbf{x})\, u_{z,i}(t) \right]^T = N_u(\mathbf{x})\,\mathbf{u}(t) \tag{5}$$

The fluid pore pressure and Darcy's flux are discretized in space with piecewise constant polynomials and in the lowest order Raviart-Thomas space [7], respectively. Denoting by $n_e$ and $n_f$ the number of elements and faces, respectively, $p$ and $\mathbf{v}$ are approximated as:

$$p(\mathbf{x},t) \simeq \sum_{j=1}^{n_e} h_j(\mathbf{x})\, p_j(t) = \mathbf{h}^T(\mathbf{x})\,\mathbf{p}(t), \quad h_j(\mathbf{x}) = \begin{cases} 1 & , \ \mathbf{x} \in T^{(j)} \\ 0 & , \ \mathbf{x} \in \Omega \setminus T^{(j)} \end{cases} \tag{6}$$

$$\mathbf{v}(\mathbf{x},t) \simeq \sum_{k=1}^{n_f} \mathbf{w}_k(\mathbf{x})\, q_k(t) = W(\mathbf{x})\,\mathbf{q}(t), \quad \mathbf{w}_k(\mathbf{x}) = \begin{cases} \pm \dfrac{(\mathbf{x}-\mathbf{x_k})}{3\left|V\left(T^{(j)}\right)\right|} & , \ \mathbf{x} \in T^{(j)} \\ \mathbf{0} & , \ \mathbf{x} \in \Omega \setminus T^{(j)} \end{cases} \tag{7}$$

In equations (6) and (7) $T^{(j)}$ denotes the $j$-th tetrahedron, $V$ its volume and $\mathbf{x}_k$ the position vector of the node opposite to the $k$-th face in $T^{(j)}$. The $\pm$ sign in (7) identifies a conventional face orientation such that $\mathbf{w}_k$ goes outward the element $T^{(j)}$ with the smallest index $j$. This gives rise to a unitary flux through the $k$-th face and a zero flux through all other edges. The vectors $\mathbf{u}(t)$, $\mathbf{p}(t)$ and $\mathbf{q}(t)$ whose components are the nodal displacements $u_{x,i}$, $u_{y,i}$, $u_{z,i}$, the elemental pressures $p_j$ and the edge normal fluxes $q_k$, respectively, are the discrete unknowns of the variational problem.

The governing equations are solved by the Galerkin method of weighted residuals, leading to the following semi-discrete MFE expressions of (1), (2) and (3):

$$K\mathbf{u} - Q\mathbf{p} = \mathbf{f}_1 \tag{8}$$

$$B^T\mathbf{q} + P\dot{\mathbf{p}} + Q^T\dot{\mathbf{u}} = \mathbf{f}_2 \tag{9}$$

$$A\mathbf{q} - B\mathbf{p} = \mathbf{f}_3 \tag{10}$$

where:

$$\begin{aligned} A &= \int_\Omega W^T \overline{\boldsymbol{\kappa}}^{-1} W \,\mathrm{d}\Omega & B &= \int_\Omega \boldsymbol{\omega}\mathbf{h}^T \mathrm{d}\Omega \\ K &= \int_\Omega B_u^T D_e B_u \,\mathrm{d}\Omega & P &= \int_\Omega \phi\beta\mathbf{h}\mathbf{h}^T \,\mathrm{d}\Omega \\ Q &= \int_\Omega \alpha B_u^T \mathbf{i}\mathbf{h}^T \,\mathrm{d}\Omega & \mathbf{f}_1 &= \int_\Omega N_u^T \mathbf{b} \,\mathrm{d}\Omega + \int_{\Gamma_N} N_u^T \mathbf{t}_N \,\mathrm{d}\Gamma \\ \mathbf{f}_2 &= \int_\Omega \mathbf{h} f \,\mathrm{d}\Omega & \mathbf{f}_3 &= -\int_{\Gamma_p} p_D W^T \mathbf{n} \,\mathrm{d}\Gamma \end{aligned} \tag{11}$$

with $B_u$ and $D_e$ the strain-displacement and the elastic moduli matrix, respectively, $\mathbf{i}$ the Kronecker delta in vectorial form, and $\boldsymbol{\omega}$ a vector whose components are equal to $\mathrm{div}(\mathbf{w}_k)$, $k = 1, \ldots, n_f$.

The system of differential-algebraic equations (8), (9) and (10) is numerically integrated in time by a finite difference scheme. Consider any time-dependent function to vary linearly in time between $t$ and $t + \Delta t$, and approximate any time-derivative at the intermediate instant $\tau = \theta\,(t + \Delta t) + (1 - \theta)\,t$ with a simple incremental ratio ($\theta$ is a scalar value comprised between 0 and 1). Setting $\gamma = \theta \Delta t$ and $\psi = (1 - \theta)/\theta$, the numerical solution at time $t + \Delta t$ can be computed by solving the linear algebraic system:

$$\mathcal{A}\mathbf{z}^{t+\Delta t} = \mathbf{f}^t \tag{12}$$

where:

$$\mathcal{A} = \begin{bmatrix} P & Q^T & \gamma B^T \\ Q & -K & 0 \\ \gamma B & 0 & -\gamma A \end{bmatrix} \qquad \mathbf{z}^{t+\Delta t} = \begin{bmatrix} \mathbf{p}^{t+\Delta t} \\ \mathbf{u}^{t+\Delta t} \\ \mathbf{q}^{t+\Delta t} \end{bmatrix} \qquad \mathbf{f}^t = \begin{bmatrix} \mathbf{f}^{(p)} \\ \mathbf{f}^{(u)} \\ \mathbf{f}^{(q)} \end{bmatrix} \tag{13}$$

$$\mathbf{f}^{(p)} = (\Delta t - \gamma)\left[\mathbf{f}_2^t - B^T \mathbf{q}^t\right] + Q^T \mathbf{u}^t + P\mathbf{p}^t + \gamma \mathbf{f}_2^{t+\Delta t} \tag{14}$$

$$\mathbf{f}^{(u)} = \psi\left[K\mathbf{u}^t - Q\mathbf{p}^t - \mathbf{f}_1^t\right] - \mathbf{f}_1^{t+\Delta t} \tag{15}$$

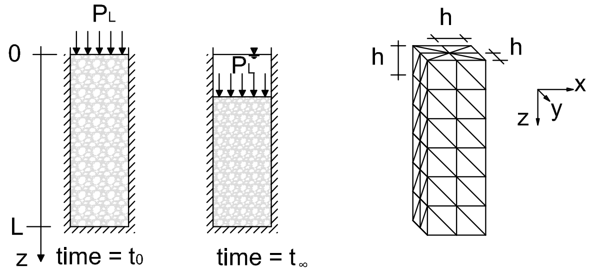
$$\mathbf{f}^{(q)} = (\Delta t - \gamma)\left[A\mathbf{q}^t - B\mathbf{p}^t - \mathbf{f}_3^t\right] - \gamma \mathbf{f}_3^{t+\Delta t} \tag{16}$$

The matrix $\mathcal{A}$ in (12) has size $n_e + 3n_n + n_f$ and is sparse, symmetric and indefinite. Suitable solvers for (12) belong to the class of the iterative projection-type Krylov subspace methods properly preconditioned. The explicit construction of $\mathcal{A}$, however, is generally not convenient from a computational point of view. In fact, while $A$, $B$, $K$, $P$ and $Q$ can be computed just once at the beginning of the simulation as they do not depend on $t$, $\mathcal{A}$ changes at each step because $\Delta t$, hence $\gamma$, is generally increased as the consolidation proceeds. Therefore a specific block version of a preconditioned Krylov subspace method proves appropriate. As to the preconditioner, we develop a variant of the block constraint approach successfully applied to standard FE consolidation models, e.g. [8], in order to accelerate the Symmetric Quasi-Minimal Residual solver [9] which has proved a robust and efficient algorithm for sparse symmetric indefinite problems, e.g. [10]. The resulting algorithm for the solution of equations (12) is provided in detail in [11].

## 3 NUMERICAL RESULTS

### 3.1 Model verification

The model is verified against Terzaghi's classical consolidation problem, consisting of a fluid-saturated column of height $L$ with a constant loading $P_L$ on top (Figure 1). Drainage is allowed for through the upper boundary only. The basement is fixed. The




| | | |
|---|---|---|
| $k$ | [m/s] | $10^{-5}$ |
| $\phi$ | [-] | 0.375 |
| $\beta$ | [MPa$^{-1}$] | $4.4 \times 10^{-4}$ |
| $\lambda$ | [MPa] | 40.0 |
| $\mu$ | [MPa] | 40.0 |
| $\alpha$ | [-] | 1.0 |

Figure 1: Sketch of the setup and hydro-mechanical parameters used in Terzaghi's consolidation test.

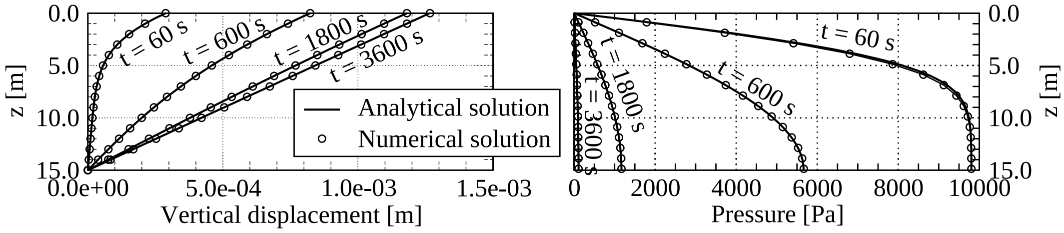
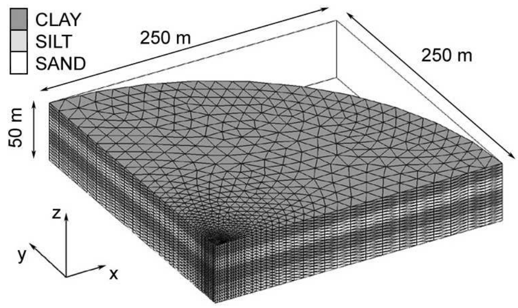



Figure 2: Terzaghi's problem: analytical and numerical solutions for the vertical displacement (left) and the pore pressure (left), with $h = 0.5$ m, $\Delta$t = 0.1 s and $\theta = 1$.

load is applied istantaneously at time $t = 0$. A homogeneous sandy column with unit section and $L = 15$ m is simulated, with the relevant hydraulic and mechanical properties summarized in Figure 1. The prescribed distributed load $P_L$ is $10^4$ Pa. The column is discretized into regular tetrahedrals with a characteristic element size $h = 0.5$ m (Figure 1). The time integration is performed with a first-order implicit scheme ($\theta = 1$) and a constant time step $\Delta t = 0.1$ s. The simulation proceeds until steady state conditions are attained. A good matching between the analytical, e.g. [12], and the numerical solution is obtained for both vertical displacement and pore pressure, as is shown in Figure 2. The convergence properties of the proposed approach are discussed in [11].

### 3.2 Realistic applications

The 3-D MFE model has been experimented with in two realistic applications addressing the consolidation of a shallow formation in the geological basin underlying the Venice lagoon, Italy. A cylindrical stratified porous volume made of a sequence of alternating sandy, silty and clayey layers down to 50 m depth is simulated. The hydro-geological properties are summarized in Figure 3 and are representative of a shallow sedimentary sequence of the upper Adriatic. The axial symmetry of the model geometry allows for the discretization of one fourth only of the overall porous volume (Figure 3) with zero flux and horizontal displacement prescribed on the inner boundaries. The following additional boundary conditions apply: the outer boundary is fixed and drained, the bottom is fixed and impervious, the top is traction-free and drained. As shown in Figure 3, a vertically




| | | Clay | Silt | Sand |
|---|---|---|---|---|
| $k_{xx} = k_{yy}$ | [m/s] | $10^{-7}$ | $10^{-6}$ | $10^{-4}$ |
| $k_{zz}/k_{xx}$ | [-] | 1 | 1 | 0.1 |
| $\phi$ | [-] | 0.3 | 0.3 | 0.3 |
| $\beta$ | [MPa$^{-1}$] | 4.32e-4 | 4.32e-4 | 4.32e-4 |
| $\lambda$ | [MPa] | 1.227 | 1.227 | 2.597 |
| $\mu$ | [MPa] | 0.819 | 0.819 | 1.732 |
| $\alpha$ | [-] | 1.0 | 1.0 | 1.0 |

Figure 3: Axonometric view of the FE grid (left) and hydro-geological properties of the shallow sediments in the Upper Adriatic basin (right) used in the realistic model applications.

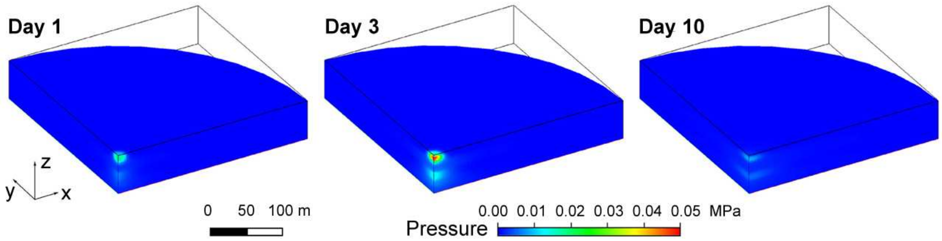



Figure 4: Test case 1: pore pressure variation vs. time due to the application of a surface load.

regularly refined grid totaling $n_n = 13,356$, $n_e = 70,080$ and $n_f = 143,368$ is used with an overall model size equal to 253,516.

### 3.2.1 Test case 1: surface load

A uniform surface load distributed over a circular area centered on the domain top with a 10-m radius is applied. The load is set equal to 8 kN/m$^2$. The load is assumed to increase linearly from 0 to 8 kN/m$^2$ within 3 days and then to remain constant. As the first layer consists of low permeable sediments, the pore pressure is expected to initially rise at the load application as a consequence of the almost undrained deformation of the clay. The overpressure gradually dissipates in time, with the dissipation rate depending on sediment transmissivity. This is physically related to the zero volume change rate prescribed at the initial time for the porous medium, which represents the main source of instability in the numerical pore pressure calculation. Moreover, the induced overpressure is generally pretty small, so reproducing it numerically may be a difficult task. The overpressure rise and dissipation in time as simulated by the MFE coupled model are shown in Figure 4. Despite the small overpressure, no oscillations in the numerical solution are observed. As the pore water flows out of the top draining surface the soil consolidates and the ground surface subsides.

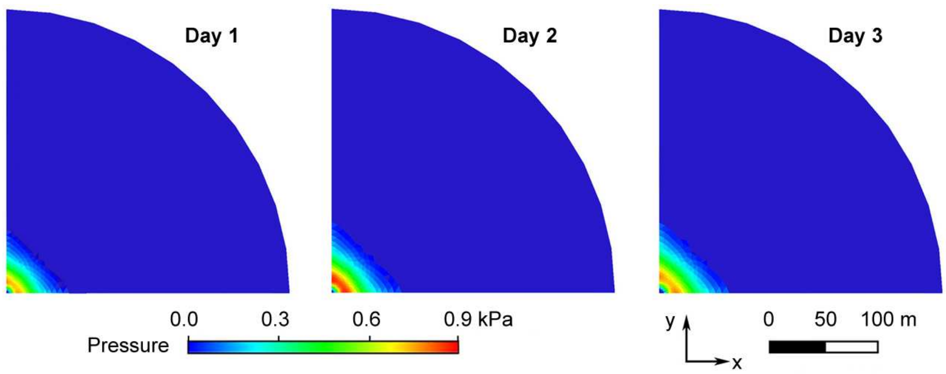



Figure 5: Test case 2: pore overpressure vs. time on a horizontal plane located in the middle of the upper clay layer.

### 3.2.2 Test case 2: the Noordbergum effect

One of the most interesting physical processes accounted for by coupling between fluid flow and soil stress is the pressure rise occurring in a low permeable layer confining a pumped formation [12]. The phenomenon is called Noordbergum effect by the name of the village in The Netherlands where it was first observed. Because of the small overpressure involved, especially when pumping occurs at a shallow depth, the Noordbergum effect is quite difficult to simulate numerically in a stable way. A constant withdrawal rate of 8 l/s is prescribed from the shallowest sandy layer (Figure 3) through a vertical well located at the centre of the simulated cylindrical porous volume. The pore pressure in the pumped formation achieves a maximum drawdown of 0.15 MPa 20 days after the beginning of pumping. To reveal the Noordbergum effect Figure 5 provides the numerical pore pressure solution as obtained in a 3-m deep horizontal plane, i.e. in the middle of the upper clay layer. The pore pressure increases at the initial stage of pumping with a very small value (about 1 kPa, i.e. more than 100 times smaller than the largest drawdown), then quickly dissipates as the consolidation proceeds. The numerical solution appears to be stable with no oscillations and a good degree of symmetry.

## 4 CONCLUSIONS

A fully coupled 3D MFE model for the simulation of Biot consolidation has been developed with the aim at alleviating the oscillations of the pore pressure at the initial stage of the process as predicted by traditional FEs. A linear piecewise polynomial and the lowest order Raviart-Thomas mixed space are selected to approximate the medium displacement and the fluid flow rate, respectively, thus ensuring an element-wise mass conservative formulation and preserving meanwhile the practical advantage of using low-order interpolation elements. A finite difference scheme is used for integration in time. The numerical solution at each time step is obtained with an ad hoc algorithm that takes advantage of the block structure of the algebraic linearized system, addressing the problem in a fully coupled way. The model is verified against the well-known Terzaghi's analytical solution and successfully experimented with in realistic large-size complex problems, such as the consolidation of low permeable layers due to a sudden load and the Noordbergum effect, with the generation of no instabilities in the pore pressure prediction.

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